DOCUMENT RESUME

ED 059 877 SE 012 735

TITLE	Tentative Outlines of a Mathematics Curriculum for Grades 7, 8, and 9. SMSG Working Paper.
INSTITUTION	Stanford Univ., Calif. School Mathematics Study Group.
PUB DATE	Jul 66
NOTE	504p.
EDRS PRICE	MF-$0.65 HC-$19.74
DESCRIPTORS	Algebra; *Conference Reports; *Curriculum; Curriculum Development; Geometry; Mathematical Applications; Mathematical Models; *Secondary School Mathematics; *Textbook Preparation
IDENTIFIERS	*School Mathematics Study Group

ABSTRACT

This document is the report of a curriculum writing session. Using the recommendations and suggestions of the planning conference (SE 012 733), the writers produced detailed outlines for the grade seven through nine curriculum. The purpose and rationale of each unit are stated, and the separate sections outlined. Sample exercises are included for most units. The aims of making mathematics more relevant to real world problems, and of avoiding excessive abstraction and rigor, are continually stressed. Also included are three general reports on geometry, real analysis, and mathematical models; several position papers on specific topics; and a comparison of the new syllabus with the "first round" curriculum. (MM)

U.S. DEPARTMENT OF HEALTH,
EDUCATION & WELFARE
OFFICE OF EDUCATION
THIS DOCUMENT HAS BEEN REPRODUCED EXACTLY AS RECEIVED FROM THE PERSON OR ORGANIZATION ORIGINATING IT. POINTS OF VIEW OR OPINIONS STATED DO NOT NECESSARILY REPRESENT OFFICIAL OFFICE OF EDUCATION POSITION OR POLICY

SCHOOL MATHEMATICS
STUDY GROUP

ED 059877

TENTATIVE OUTLINES
OF A
MATHEMATICS CURRICULUM
FOR
GRADES 7, 8, AND 9

SMSG Working Paper

July, 1966

SE 012 735

CONTENTS

Preface . . . .
List of Participants . . . .
Report of the Geometry Committee: The Philosophy of the Geometry Program 1
Report of the Real Analysis Committee: The Outlining Principles of the Real Analysis Group 4
Report of the Modeling Committee 12
Contents of Grade 7 31
Outlines of Grade 7 Chapters 41
Contents of Grade 8 176
Outlines of Grade 8 Chapters 184
Contents of Grade 9 320
Outlines of Grade 9 Chapters 324
Summary outline, Grades 7-9 381
Comparison of new sequence with the original SMSG sequence for Grades 7-9 386
On "Applications" by Clyde Corcoran 389
Some Comments on the Role of Flow Charting in Junior High School Mathematics by Sidney Sharron 392
Probability and Statistics for Grades 7-9 by Richard Dean and Martha Zelinka 417
Vectors on a Line by Hassler Whitney 424
Outline - Vectors by H. S. Moredock and W. H. Sandman . . . 434
Restricting and Freeing the Intuition by Henry O. Pollak . 468
On the Introduction of Mathematical Concepts by Hassler Whitney 470
The Role of Logic in Elementary Mathematics by Hassler Whitney 479
The Use and Importance of Definition in Mathematics by Henry O. Pollak 485
Uninvited Comments on the Definition of Function by Gail S. Young 487
On the Setting and Function of Sets and Functions by Leonard Gillman 490
On "On the Setting and Function of Sets and Functions" by Gail S. Young 494

PREFACE

During a four-week session which started June 27, 1966, a team of mathematicians and mathematics teachers formulated preliminary recommendations for the curricular experimentation which SMSG plans to carry on during the next few years. A list of the participants in this session follows.

The recommendations take the form of detailed outlines of most chapters of a mathematics program for grades seven through nine. These will provide a framework for the experimental writing to be carried out during the coming academic year.

At the opening plenary session the participants reviewed the recommendations of the New Orleans Conference and agreed to outline a curriculum for grades seven through nine which would be in general agreement with the New Orleans recommendations. Three subcommittees were formed. The first was asked to consider further the role of mathematical models in this new curriculum. This committee submitted its report at the end of the first week, and the members of the committee were co-opted into the other two committees or undertook special assignments.

The second subcommittee was asked to consider the topics in geometry that should be incorporated in the curriculum for grades seven through nine, keeping in mind that for those students taking more than three years of secondary school mathematics the tenth grade course would probably include at least a semester of formal synthetic geometry. The third subcommittee was asked to consider the topics in arithmetic and algebra to be included in the seven through nine curriculum.

The latter two subcommittees met together from time to time to discuss the meshing together of the geometric and the algebraic sequences.

Occasional plenary sessions were held so that the entire group could discuss the materials which had been produced. At a final plenary session the general reports of the geometry and of the arithmetic-algebra subcommittees were received, and plans for the next steps in the over-all project were discussed.

The two above mentioned general reports, together with the report of the committee on mathematical models, are followed in this document by a topical outline for Grade 7. This in turn is followed by detailed outlines of most chapters for Grade 7. Grades 8 and 9 are treated similiarly.

The next section of the report is designed to facilitate comparisons between the present SMSG program and the proposed new one. A summary outline of the new program for Grades 7 through 9 is followed by a list of chapter headings from the present SMSG books for these three grades. In this latter the location of the corresponding material in the new sequence is indicated.

The report concludes with a number of papers of a general nature which were prepared during the session. Included also is outline of a unit on vectors which had been prepared in the summer of 1965 at the request of the SMSG Panel on Science for consideration in the new curriculum.

LIST OF PARTICIPANTS

Mrs. Pamela Ames - University of Chicago
Max S. Bell - University of Chicago
Jean M. Calloway - Kalamazoo College
Clyde L. Corcoran - California High School, Whittier
Richard A. Dean - California Institute of Technology
W. Eugene Ferguson - Newton High School, Newtonville, Massachusetts
Leonard Gillman - University of Rochester
Richard A. Good - University of Maryland
Miss Lenore S. John - University of Chicago
Mario L. Juncosa - The Rand Corporation
Lowell J. Paige - University of California at Los Angeles
Max Peters - Wingate High School, Brooklyn, New York
Henry O. Pollak - Bell Telephone Laboratories
Walter Prenowitz - Brooklyn College
Mrs. Persis O. Redgrave - The Norwich Free Academy, Norwich, Conn.
Sidney Sharron - Los Angeles City School District
Henry W. Syer - Kent School, Kent, Connecticut
Hassler Whitney - Institute for Advanced Study, Princeton
Miss Martha Zelinka - Weston High School, Weston, Massachusetts

The following individuals attended most of the plenary sessions:

William G. Chinn - School Mathematics Study Group
Warren Stenberg - University of Minnesota
Gail S. Young - Tulane University

REPORT OF THE GEOMETRY COMMITTEE

The Philosophy of the Geometry Program

I. How is Geometry to be Conceived?

Most of our knowledge (certainly our scientific knowledge) refers to or is set in the framework of physical space. When a child begins to crawl, he discovers geometric properties of space by pressing against walls, by patiently putting things in cupboards and just as patiently taking them out, by finding paths which take him back to where he started. This knowledge, gained concretely and intuitively over the years, is in a conventional treatment carefully formulated and structured in a tenth grade geometry course.

We propose to approach geometry as a subject which is suggested by and modeled on our experience with physical space. Its basic concepts--for example, points, lines, segments, etc.--are suggested by objects of experience; its results can be interpreted in physical space and confirmed to a high degree of approximation. Treated in this way, geometry can become an important branch of knowledge, not just a mental exercise.

The important relation between physical space and the geometric theory we idealize or abstract from it appears twice: First in forming the concepts, since points, planes, spheres, etc., as conceived mathematically do not seem to exist in the physical world. Second, in applying geometry, for we must interpret physically or form a physical model of the concept. A point in a surveying problem may be interpreted as the overlap of two crosshairs in a telescope or in a dynamical problem as the sun or the earth.

Remark: The bearings of geometry on physical reality are important and exciting; they should be treated with judgment as opportunities arise. But they should not dominate. We are presenting a course in geometry, not a course in its application to reality.

II. How is Geometry to be Treated?

We assume two boundary conditions: (1) The student has studied the SMSG texts for grades four, five, and six (or the equivalent) and so comes to us with a nontrivial geometric experience; (2) He is to be prepared for a deductive treatment of geometry as part of a tenth grade course.

The SMSG texts for grades four, five, and six contain a rich body of geometrical material, including nonmetrical geometry, properties of geometric figures, measurement (length, area, volume), and congruence. Since we are not sure how much of this material the student is immediately able to apply, we naturally begin by employing the spiral method: We review, refine, and amplify the material already studied. Thus Chapter 1, "The Structure of Space - Nonmetrical Properties," enriches and expands the qualitative geometric knowledge gained in grades four, five, and six and introduces new ideas such as convexity and orientation. This approach is continued for congruence in Chapter 4 and for measure in Chapter 5.

Throughout the development we have the problem of presenting the subject in a concrete, intuitive, descriptive way without reducing it to a collection of more or less isolated facts. We try to take care of this by a second application of the spiral method: We focus on a new concept by concentrating on its essential features, later returning to treat its more complex aspects and its relation to other concepts. The concept of parallelism, for example, first introduced in Grade 7, Chapter 1, appears again in Grade 7, Chapter 11, "Parallelism," and Grade 8, Chapter 11, "Parallels and Perpendiculars." The concept of measure which is used throughout the course is specifically studied in four chapters that are distributed through grades seven, eight, and nine. As the preceding sentences suggest, we have chosen a dominant geometrical concept or relation as the unifying feature of each individual chapter, within which the pertinent properties are developed for all the appropriate geometrical figures to which the concept or relation is applicable.

One of the problems in teaching geometry at this level involves the quality of student understanding. Since much of the material is descriptive and concretely presented, the child may merely be developing the ability to repeat descriptions and recognize figures. We want him to comprehend properties of figures, to perceive interrelations between parts, to recognize familiar notions (e.g., congruent triangles) in a complex and unfamiliar situation. To this end we try to develop and refine his intuitive grasp of geometrical properties. (Consider how much more a topologist "sees" in a simple closed curve chalked on an inner tube than the proverbial man in the street. Or suppose a child has an intuition that a circle is round. Does he realize that is has a different roundness from a kidney bean , that it is a convex curve, or that its interior is a convex set?) One procedure we use for coping with this problem is to take a familiar figure, say, a cube, and ask the student to find in it several illustrations of an idea, for example, line perpendicular to plane, parallel lines, parallel planes, line parallel to plane, skew lines, a common perpendicular to two skew lines. This practice tends simultaneously to sharpen perception of figures and comprehension of concepts.

The treatment indicated should foster a good understanding of the concrete basis and the intuitive significance of geometric ideas. In addition, we propose to sharpen and enrich understanding of the deductive process. For this purpose we introduce many examples of deductive reasoning. These range from a one- or two-step informal proof which is not written down (the problem asks for a conclusion that requires application of one or two familiar principles in a diagram) to a deductive chain of several propositions which follow from a given set of geometric properties. A chapter on deductive reasoning is included to initiate a discussion of deductive reasoning in methematics by using examples and illustrations from algebra and geometry.

A student who has had a course of this type should be well prepared to make the adjustment to the more formal deductive treatment of geometry in grade ten. He should have assimilated a large body of geometric knowledge which rests on a concrete and intuitive basis and is partially structured by deductive proof. The problem of organizing this knowledge logically should not seem unnatural or remote to him. He certainly will not have the familiar double difficulty of trying to learn what the subject matter is about while he attempts to understand the deductive method. Is it too optimistic to hope that our program will permit an appreciable saving of time in grade ten while fostering increased understanding?

REPORT OF THE REAL ANALYSIS COMMITTEE

The Outlining Principles of the Real Analysis Group

A group of people assigned the task of developing an outline for the mathematics to be presented in the seventh, eighth, and ninth grades is faced with a most formidable problem if there are not some broad, general guidelines adopted from the very beginning. The subgroup responsible for the sequencing of the mathematical concepts related to real analysis, as opposed to geometry, met together frequently during the early days of the outlining session, and I believe the following general restrictions served as guidelines to our discussions:

(a) The background which we would assume for all students would be that found in the present SMSG texts for the fourth, fifth, and sixth grades.

(b) The mathematics to be introduced in the seventh, eighth, and ninth grades should be of value to all students as necessary for any intelligent, responsible future citizen, regardless of occupation. As a consequence of this decision reflecting a recommendation of the New Orleans planning session, we consciously replaced some special topics (modular arithmetic, finite fields) by those we felt were more appropriate, for example, the elements of probability and statistics.

(c) The work of other groups who have given considerable thought to the mathematics suitable for these grade levels should not be ignored. It did not seem reasonable for us to retrace the deliberations of the Cambridge Conference, the New Orleans planning group, etc., but to respond responsibly to their recommendations.

(d) Some of the criticism of the excessive formalism in the first round SMSG material is justified. We should keep in mind the critics' view that physical situations were not used to provide heuristic motivation for the mathematical development nor was the mathematics developed used to analyze physical problems. However, the latter complaint has been met in part by special projects of SMSG which may be used by the writing group.

(e) There must be continuity from chapter to chapter both in writing, in concept, and in depth of sophistication. These attributes in any textbook writing are so important that this may dictate placement and treatment of many topics.

When we turned to the consideration of the specific sequencing of topics for the three grades under consideration, we felt that:

(a) The concept of function should be introduced early in the seventh grade and used where appropriate but without excessive fanfare. The idea was to make the concept of function a familiar part of the student's background. Moreover, in illustrating the applications of mathematics to physical situations, it was hoped that examples could be found to show that the analysis of a graph often permits predictions which are not apparent from isolated information.

(b) Geometry and analysis must be interwoven throughout the course, each supplementing and leading the other. Graphical illustrations with coordinate systems lend clarification to many mathematical topics. (Again, in retrospect we know that there is a great deal of work remaining for the writers before this desirable integration can be realized.)

(c) The "structure" of the rational number field and the real number field should remain a unifying thread throughout the introduction of successive topics concerning rational and real numbers without excessive formalism. Some acquaintance with formal proof is desirable. Topics introduced should lead somewhere. Concerning any area, we hope that the student will eventually be able to say, "Aha, now I understand this!"

(d) In adopting a spiraling of material throughout these three grades, care must be taken to see that subject matter is used, at least in problems and hopefully in subsequent subject matter, before being studied again.

(e) Logic should be fused into the course material so that the precision of reasoning required in mathematics would gradually become accessible to all students. We felt that it was unnecessary to provide a separate chapter on truth tables, excessive formalism, or the idolatry of symbolism. (See Whitney's chapter, "On the Role of Logic in Elementary Mathematics.")

(f) Both notation and terminology introduced in these grades should be compatible with present-day usage in mathematical texts at higher levels. Here we felt perfectly at ease in accepting the recommendation of the New Orleans planning group that the "open" of sentences could be abandoned.

An initial outline of topics for real analysis, fulfilling the criteria described above, resulted in the following sequence which was agreed upon before more individual attention was given to the separate chapters.

We have separated the material into various grades as we thought appropriate.

Grade 7: Real Analysis Group

A. Graphing, Functions

We chose to begin the year by using the plotting of points in a coordinate system as a means of reviewing and extending the students' knowledge of the integers (both positive and negative) acquired in the sixth grade. This also provided a natural way to introduce the concept of a function.

B. Solutions of Simple Mathematical Sentences

The purpose of this chapter was to begin an informal discussion of "solutions" of mathematical sentences. This provided us with an opportunity to review the arithmetic operations of the positive rationals. If the rationals are to be delineated as a deductive algebraic system, then this seemed to be an appropriate place for the students to be presented with these facts. On the other hand, if we choose to extend the operations to the negative rationals, then we could not see where the negative rationals would be used for several chapters. Consequently, two versions for the introduction of the rationals have been suggested to the writing group. One version restricts its attention at this time to the positive rationals and in Chapter E motivates the extension of operations to negative rationals graphically. The other version introduces the rational field here, and Chapter E is modified accordingly.

C. Ratio, Percent, Decimals

Frankly, we succumbed to the pressures of tradition which require that these topics be included in any curriculum. We viewed this chapter as a possible means of increasing the student's arithmetic skills with nonnegative rationals.

D. Combinatorics and Probability

Here we felt that the recommendations of the New Orleans planning group required the introduction of this material, and we again had an opportunity to reinforce a student's skills with positive rationals and the notion of function. We felt that the SMSG Probability Group could provide suitable material for this chapter. We preferred to reinforce computation skills as the student studies new topics rather than through routine drill material.

E. Graphing: The Negative Rationals and Extending Arithmetic Operations to the Rational Field

We have remarked under B that the content of this chapter depends upon the approach taken there. However, at this point at the very latest, we felt that the explicit axioms for the rational number field should be displayed and discussed. This discussion should emphasize the fact that we have a deductive basis for the proofs in algebra.

F. Solutions of Systems of Mathematical Sentences

We wished to reinforce the student's skills in graphing linear functions, as well as present problems requiring computation with the negative rationals. This should include the graphing of inequalities.

G. Square Roots, Nonrepeating Decimals, Real Number System

We felt that the student should be introduced to the need of numbers beyond the rational field. Moreover, square roots were needed for the discussion of distance in the chapter to follow. We wanted the student to know that the real number system satisfied all the properties of the rational field as well as an additional axiom of completeness. We do not recommend a formal extension of the rationals to the reals, and informality should suffice.

H. Distance, Pythagorean Theorem, Circles

We are again using the material presented in the previous chapter, that is, graphing, the real numbers, etc. We thought that the analytic treatment of circles could provide the format for interesting mathematical sentences involving inequalities, for example, the set of all points outside a circle, etc.

Grade 8: Real Analysis Group

A. Exponential Function, Logarithms, Scientific Notation

We attempted to arrange material so that each year began with a topic which would involve the student in overt activities as they explored the subject matter. The construction of the logarithms table (as in the Cambridge Report) and the knowledge of a slide rule was chosen because we felt other fields would need this information about this time. Moreover, exponential functions seemed to occur in many diverse fields and should be understood. This subject matter permitted a review of exponents and factoring begun in the fifth grade. (In the final outline this material appears in the ninth grade.)

B. Measurement

The purpose of this chapter was to implement the concepts on measurement introduced in the seventh grade and to provide practice in logs, etc., of the preceding chapter. It was recommended that a statistical point of view of relative error, deviation, normal distribution of measurements be included so that we could connect these concepts with the probability introduced in the seventh grade. (In the present sequence this chapter is now in the ninth grade outline.)

C. Problem Analysis (Strategies)

It was felt that an early chapter should be devoted to developing the student's awareness of a variety of strategies for problem analysis. Since problem solving is a major activity in mathematics, whether in algebra, geometry, or applied mathematics, some early and continuous recognition must be given to the specific skills necessary to complete this activity successfully.

D. Number Theory

This chapter was designed to review and extend the "Factoring and Primes" concepts developed in grades four through six and to establish the unique factorization theorem for integers. The theorem was needed for the following chapter on the real numbers. We also wanted to use the chapter to develop some beginning concepts of logic in simple situations ("if-then," converse, negation).

E. The Real Numbers Revisited--Radicals

We felt that sometime in the eighth grade the students should again face the axiomatic nature of the real numbers and review the structure. Moreover, theorems as direct consequences of these axioms could be presented. The solution of classical problems (duplicating the cube, etc.) could lead to this reconsideration; radicals could be discussed in detail with the corresponding review of exponents and absolute value.

F. Solutions of Equivalent Mathematical Sentences

This chapter was delayed to this point in order that operations yielding nonequivalent sentences were available. We wanted to formalize the process of obtaining equivalent problems bringing logic, the structure of the real number system, and properties of order into play. We wished to be able to write precisely about systems of equations and inequalities in the next chapter.

G. Systems of Equations and Inequalities, Linear Programing

We thought this subject matter would lend itself naturally to reviewing many topics as well as introduce the student to interesting applications in modeling.

The eighth grade continuation of probability and statistics could be inserted after B, or other places might be desirable.

Grade 9: Real Analysis Group

We felt that the students' preparation was now adequate for the presentation of substantial mathematical ideas. Hence, our outline became brief, and the arrangement was tentative with considerable freedom left to the writers.

A. Quadratic Polynomials (as Functions)

This chapter could serve as an ideal place to study the translation of axes and relate these translations to the previous introduction of vectors as displacements. Likewise, the zeros of quadratic functions could lead to a treatment of factoring as needed. (In the final version this chapter appears in grade eight.)

B. Systems of Mathematical Sentences Involving Quadratic as well as Linear Functions

Here the possibility of review of factoring, radicals, solving quadratic equations is present.

C. Locus Problems

(Conic sections first round)

D. Vectors

E. Trigonometric Functions

F. Inferential Statistics

G. Complex Numbers

* * * * * * * * * *

The integration of the seventh and eighth grade geometry and real analysis changed the order of some of the materials, particularly in the eighth grade. Specifically, we dropped exponents and logarithms while substituting quadratic functions of grade nine to appear later in the year. The measure theory was combined with the geometric subject matter and moved to grade nine. These decisions may need review.

Using geometry as a process of "modeling" the real world, we saw that this could lead naturally to the need for coordinates as necessary to provide more "local" information in physical problems. Hence there appeared to be sufficient reason to start the seventh grade with a chapter on geometry and to follow this immediately with A of our outline for grade seven. Thereafter the blending of geometry and the real analysis suffered from the demand that chapter outlines had to be produced simultaneously. Moreover, the real analysis group thought that it had to consider in greater detail several questions which are not reflected in the outline. Some of these specific questions were:

1. Notation for Functions. We believe that the introduction to functions should be quite informal with examples to illustrate that the student has used the idea for quite some time. We finally agreed that the notation

$$f:\ x \longrightarrow 2x + 3\ ,$$

as opposed to ordered pairs, was best for this grade level. It seemed to us that even the generality

$$f: \; x \longrightarrow f(x)$$

could be postponed by a judicious choice of functions.

2. Variables. We feel that this topic has been subject to so much discussion during the past ten years that we need not add further confusion. Our position is simply "Avoid the use of the term variable as a mathematical entity at this grade level." It is hoped that the written material will make clear the role of the variables used.

3. A Motivation of Negative Multiplication by Means of Vectors on a Line. Professor H. Whitney presented an outline whereby the definition of multiplication of negative numbers becomes a consequence of the study of a one-dimensional vector space along the number line. However, it was not clear to some of us how certain difficulties relating the "scalars" to the "vectors" are to be avoided. Further investigation of this approach is certainly warranted, especially since displacements are to be introduced in space in the eighth grade.

4. Polynomials. The usual discussion regarding forms, expressions, or functions flourished again. We decided that we could restrict our attention to quadratic functions and avoid for the time being the distinctions which invariably arise. We did not believe that it was necessary for everyone to know the theory of polynomial rings.

Finally, in detailing the outlines for the chapters we have described, it appears that we have lost sight of some of the general criteria which we set for ourselves. We urge the writers to return to this document whenever the details have obscured the attitude we sought to impart.

REPORT OF THE MODELING COMMITTEE

In this report we are attempting to show the place of modeling in mathematics instruction and how to implement it throughout the sequence of grades seven, eight, and nine. (For some philosophical remarks on modeling, see Appendix A.)

I A. "Mathematical model" (noun) and "modeling" (verb) should be brought into the picture early in the seven-nine sequence and should be a thread worked in throughout the sequence, since every application of mathematics to the real world involves a model. Some models are by now implicit (most of the time we are not aware of the model), while others are explicit and need to be worked out in detail.

B. As part of any first chapter in the seventh grade book, we should begin to talk about mathematical modeling. (See Appendix B1.)

C. At an appropriate place in each succeeding section of material (on geometry, probability, etc.), bring in "modeling" again along with some examples for which the subject matter at hand provides mathematical models.

D. Later when more mathematical maturity has been gained by the student, more emphasis in the form of a specific chapter may be given to modeling.

II A. For the teacher, provide clear expositions of what is meant by mathematical models and modeling. Give the teacher a feeling for the goal to be achieved by the persistent thread on modeling that runs through the three-year sequence. (See Appendix B2.)

B. For students, provide a variety of examples (see III A3). Some of these should carry through in a spiral fashion. (See some of the examples in Appendix C.)

C. Lead the student to appreciate the hard work and extreme care which must go into mathematical models of great complexity, such as those that permit man to place a capsule on the moon. He should then be moved to exhibit the same care, hard work, and attention when using mathematics in solving a problem within his power.

D. Aim for frequent reinforcement and for a wide variety of experiences that will eventually illustrate all the important features of mathematical modeling.

E. In addition to the examples that come up as part of the text, supply material of the following sort: (1) longer expository articles or "feature" films in an "inspiration-guidance" mold on uses of mathematics and mathematical models in various fields; (2) short film clips that either provide data for a modeling sequence or illustrate such a sequence.

III A. Features of Good Modeling Examples

1. Real life situations must be real (that is, not phony), interesting, and must contain a question for which the answer is not obvious or trivial. The solution in the mathematical model ought to be capable of interpretation and testing in the real life situation.

2. The mathematical model needed to analyze the situation and reformulate the question in mathematical terms should not be trivial or implicit in the description of the situation.

3. For some examples the mathematical manipulations should be within the capabilities of the students at the time the example is introduced. For others the model may call for new techniques, and the model can serve as motivation for the introduction of new mathematics. For still others (not too large a number), the skills needed could be well beyond the capabilities of the student, but nevertheless, the problem can serve as motivation for the continued study of mathematics.

4. Examples should be devised so that the models will have to be constructed by omitting or ignoring many details in the real situation, and this selectivity in devising the first model and successive models should result in approximations, some of which are good for one purpose and others better for slightly different purposes.

5. It should be emphasized that any model is only an attempt to represent certain aspects of the situation which are important for particular restricted purposes.

6. The purpose of constructing the mathematical model is to clarify relationships so as to exhibit clearly the important features of the situation and contribute to answering questions which could not be

answered easily without the model, i.e., analysis with the ultimate objective of prediction.

B. Things to Watch in Introducing Modeling

1. Avoid sloganeering.

2. Don't model the student to death (as with sets).

3. Use a variety of nomenclature, since words have different connotations in the physical world.

4. Get some examples in which the situation or concepts are abstract and the model is concrete; e.g., a model for the real numbers is the number line which we draw.

5. Achieve a proper balance in introducing models. Don't give the impression that mathematics exists only because of its applications. On the other hand, remember that only a small fraction of students using the texts are going to be mathematicians.

6. Remember the need to revisit the real world frequently during a course, not just at the beginning.

7. Get examples in which complete reliance on the physical model or on intuition leads us astray, whereas the mathematical model may lead to "truth" uncontaminated by the prejudice of physical experience or "common sense."

8. Remember the need for a careful selection of workable examples; others may better be left for general remarks about applications.

APPENDIX A

Philosophical Remarks on Model-Making

by M. L. Juncosa

Pervading almost every element of the set of human intellectual activities from the beginning has been a search for an "understanding of phenomena." This holds for existential, physical, sociological, conceptual as well as concrete phenomena; it matters not which. This pursuit of causation and knowledge of structure can be motivated by desire for comfort, fear of the unknown, satisfaction of curiosity, etc. One finds this in theories on the origins of primitive religions and magic.

For the scientist--and this includes mathematicians--a strong motivation is the desire for predictability; that is, within certain bounds the structure or model can be interpreted as being "consistent." For the inductive scientist, results of the theory, i.e., predictions, will "agree" with experiments. For the mathematician, contradictory theorems will not result.

To arrive at conclusions, a process of what some people call model building is engaged in. The primitive man invents concepts of supernatural gods with anthropormorphic attributes, such as anger at broken taboos, and enormous powers, such as the power to cause awe-inspiring meteorological phenomena.

The scientist observes physical, economic, sociological, biological, or psychological phenomena; he invents an idealization of them according to some laws which may exist from previously studied "similar" (maybe even "isomorphic") situations or which he constructs ad hoc, containing what is felt is the "essence" of the observations; then as what he calls a logical consequence of these laws, he makes certain statements or predictions, asserting that he has now an explanation, a theory, or more modestly, a model (not necessarily unique and which may or may not be mathematical) for the phenomenon.

The mathematician constructs many conceptual models of other concepts or theories that he is in the process of exploring. He frequently finds that in the model or image he may have greater insight or may be able to use language which is not quite available for the original. This enables an "end-run" in the proofs of goal theorems or suggests new goal theorems and techniques for the original. It is not uncommon that he makes physical models and pictures as models for his theory for greater elucidation and inspiration.

It is essential to recognize the universality and variety in the philosophy of modeling, regardless of what it is called. Not only does everyone do modeling at some intellectual level, but frequently transitions from one world to another and back again are made. An engineer may make a mechanical model (here called analog) of springs, weights, and dash-pots for an electrical circuit of resistances, capacitances, and inductances, or vice versa. And if he has a mathematical model as well, he may not even make, i.e., physically construct, the mechanical or the electrical analog but will rather solve the pertinent equations interpreting the results for either of the physical situations, talking in the isomorphic language of the one most familiar to him, even though he may be solving a problem concerning the other primarily because "it is easier (for him) to see it that way."

(We like to point out that usage of this word "model" differs markedly. We are using "model" as a copy, picture, image, representation, formulation, etc., of the original, as in the usage where Klein and Poincare models are examples of non-Euclidean geometries. This is in contradistinction to the use of the word in connection with an artist's or photographer's subject where the original is the model. No strong preference is expressed here, but we chose the usage in this work because of the confirmed usage in many applied mathematical circles.)

Returning to the variety of instances of the practices, we have

1. the process of going from the real world to the real world, cited above (construction of analog computer, slide rules, etc.);

2. the process of going from the real to the conceptual, mathematical, and then back to the real (mathematical physics, mathematical biology, mathematical economics, operations research, applied mathematics

in general, the process being admirably described in Burrington's article,[1] "On the Nature of Applied Mathematics");

3. the process of going from the conceptual to the real and back again (construction of Venn diagrams and switching circuits for set theoretic and Boolean operations, construction of finite group multiplication tables, construction of rings, trees, graphs, knots, cross caps, Klein bottles for certain topological objects);

4. the process of going from a conceptual to another conceptual without passing to the real and back again (identification between real numbers and points on a line, language structures as trees or graphs);

5. and even processes of going from lower to higher conceptual levels and back, as well as vice versa.

Some model-making goes from deterministic conceptions to probabilistic ones and back, as exemplified by solving either the heat equation or the potential equation by random walks, which is a special case of the so-called Monte Carlo method. Polya's model of contagion is another example.

Since the criticisms of the previously constructed curriculum included the insufficient liaison with the physical world, we take as precepts the goals of R. C. Buck's article,[2] in particular goal No. 1, and to a lesser extent goals Nos. 2, 3, and 6, as having relevance to the question of modeling. And in particular, we further restrict attention to the philosophy as applied to real world problems. While the term has not appeared in many parts of mathematics in the past, the term "mathematical models" is used very extensively in biology, economics, management psychology, operations research, control applications, chemistry, statistical mechanics, etc., where work has only recently been "mathematized." Thus we accept this usage, recognizing that modeling is a broader intellectual concept and cautioning strongly against a monopoly on the use of the term to avoid polarization of attitudes among people who should recognize the universality of the process. Thus many words have been used and with different connotations. They should be kept and used appropriately with the purpose of the particular choice being held in mind. Synonyms should always be pointed out with their slightly different shades of connotation indicated.

---

[1] R. S. Burrington, "On the Nature of Applied Mathematics," American Mathematical Monthly, April, 1949. See Appendix D3.

[2] R. C. Buck, "Goals for Mathematics Instruction," American Mathematical Monthly, November, 1965. See Appendix D2.

Another observation on models is the strong essence of approximation present, particularly in real world problems involving either continuous variables or large numbers of variables (e.g., gases, populations, traffic) in some problems. We have simple examples of this in the representation (model) of a flat sheet of paper as a rectangle for most purposes but as a rectangular parallelpiped (!) when one is interested in estimating the volume of a book; the habitable world in antiquity or much smaller localities today as a flat segment of a plane (ignoring the local mountains, valleys, rivers) but the habitable world as a sphere today (or happily for Eratosthenes seeking an estimate of the size of the earth), or the earth as oblate spheroid for satellite work because of the precision required in orbit computation; the circulatory system as a pump; a gas (collection of molecules) as a fluid; etc. These approximations frequently are made to enable the recognition of mass behavior or macroscopic behavior; at other times they are made to make a problem either mathematically tractable or computationally feasible.

An essential factor in a good model in this class of situations is that of stability; small deviations in the original should result in small variations in the predicted result. In other words, the conceptual transformation from the real world of observations through the model, through the mathematical operations and back to the real world prediction is continuous with respect to the appropriate norms. A poor model in these situations is one with enormous variations in the results for small deviations in inputs. Parenthetically, we should observe that occasionally it is because of the nature of some startling variation in the predictions by inadequate models that original discoveries are made by entirely new formulations.

In another class of models the essence of approximations does not figure strongly or even at all. In these it is structure that is important: Do the variables in the problem figure linearly or not? Can an algebraic group structure be assumed in the model for the phenomenon or not? Is the model for the world Euclidean (parallel postulate, etc.) or not? Problems for which tree-like or graph-like models are constructed have this flavor. It is important to recognize that the notion of stability seems to be irrelevant here. The familiar problem of the three houses desiring three utilities without overlapping connections from the mains to the homes, modeled as an attempt to construct a certain (impossible) graph of six vertices, is again structural. Stability and approximation considerations are irrelevant, there being no "neighboring" problem. (The "solution" is "possible" as soon as the number of homes is reduced to two.) Perhaps it is worthwhile to discourse on these differences.

APPENDIX B

1. Suggestions for initial introduction to modeling in the first chapter of the seventh grade book

Start with examples of the uses of models (not necessarily mathematical) of real world phenomena and objects. Some of these may be physical, for example, clay models of animals, and others conceptual, such as the plane or the sphere as models of the earth. Point out some of the obvious features of such copies of real phenomena. Indicate the special way in which we will use the word "model"; i.e., for us if the artist is painting a portrait, it is the picture, not the lady, that is the model. Following this, remark that mathematical models are characterized by the use of ideas represented by mathematical symbols in constructing the model, rather than clay, paint, or strictly verbal description.

2. A sample of material on modeling which might be included in the teacher's manual

The purpose of this topic is to develop the idea of modeling as a means by which mathematics is related to problems which arise in the real world. At times the process should be made explicit so that the student can see the following: the real life situation translated into a mathematical model, the mathematical manipulation within the mathematical model, the obtaining of a result, its interpretation back in the life situation, and the testing of the validity of the result and therefore of the adequacy of the model. This process should be emphasized at appropriate points throughout the texts.

The word model is not strange to the students. Model cars, model airplanes, and models of most objects are part of his environment. However, the philosophical connotations associated with mathematical modeling--that of providing an interaction between mathematics and reality--is likely to be a new chanel of thought for a seventh grade student.

A general discussion with the class focused on the various reasons for constructing models could direct the students' attention towards such practical aspects as size, location, ease of handling, highlighting, and opportunity for further study. It is suggested that encouragement be given to the student to think up synonyms for models, such as picture, likeness, illustration, image, copy, representation, replica, reflection, pattern, resemblance, and facsimile. In addition to enhancing the idea of modeling, the synonyms give rise to a conjecture concerning the degree of likeness one may expect or desire in a model. Of course at best a model is somewhat less than the original. However, the validity of a model is related to its usefulness or how well it does what it was intended to do.

It is intended that this discussion will converge on models of real world objects or situations that are formulated by mathematics. At this point a problem within the frame of reference of a seventh grade student should be offered to illustrate the facility that a mathematical model offers in predicting a solution.

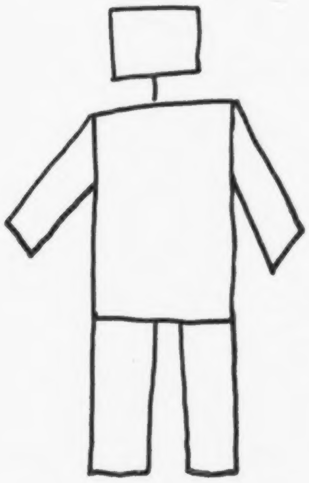
APPENDIX C

Examples of Modeling Suggested During our Discussions

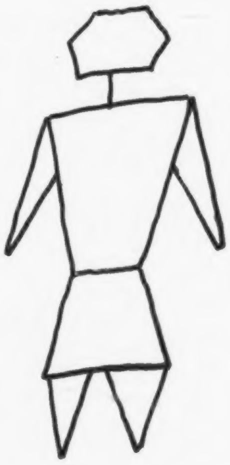
1. Find the volume of the body of a man six feet tall. Suppose the average thickness of his head is 7 inches, his arms 3 inches, his truck 10 inches, his legs 4 inches.

Possible models suggested might be:

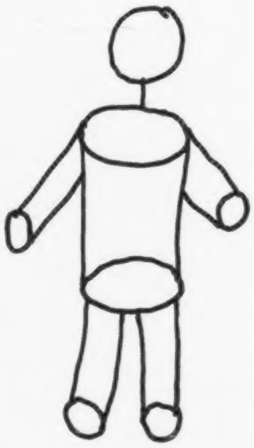
Model 1



Model 2



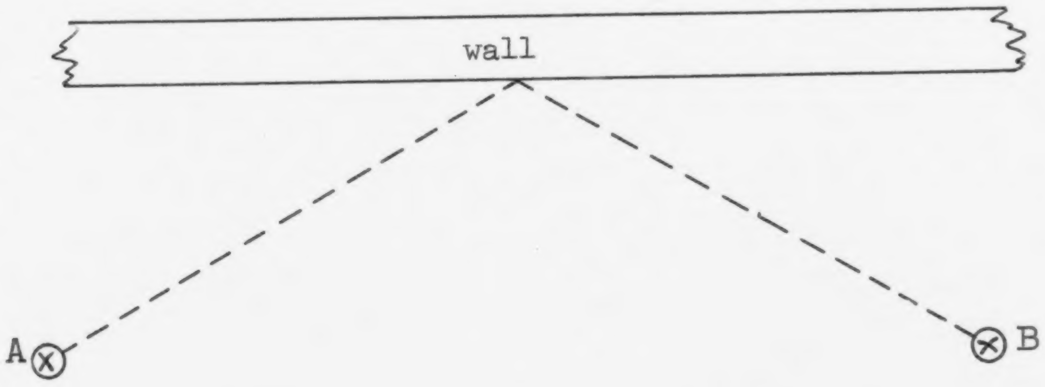
Model 3



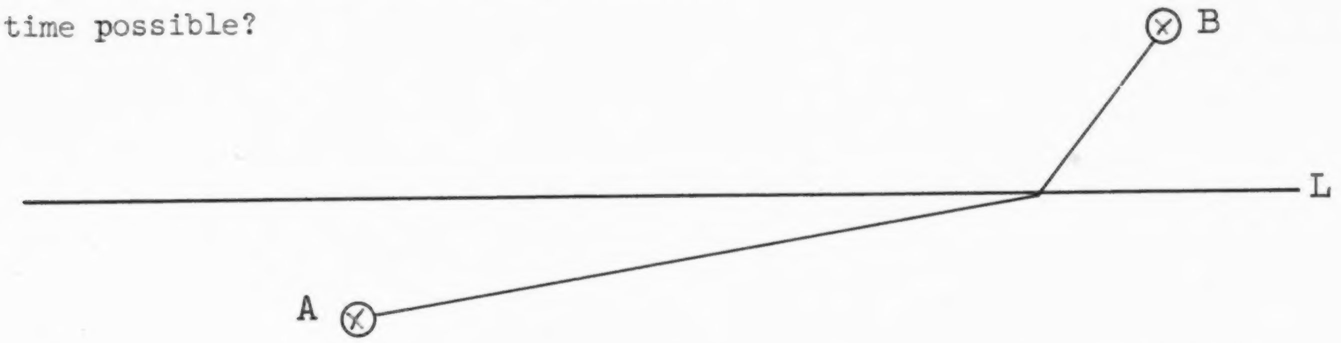
Test the accuracy of the students' models by submerging them in water and calculating the change in height of the water.

2. Measuring the earth as in "How Far is it From Here to There?" Mathematics Teacher (Feb. 1965), 57:123-130, by I. Fisher.

3. Reflection example. Idea: begin at A, touch the wall, and arrive at B in as short a time as possible.



4. Refraction example. Idea: Run from A to line L at r ft/sec., and walk from there to B at w ft/sec. To what point on line L should I head in order to reach B in the shortest time possible?



Second time: Use the problem of apparent change of direction of a stick immersed in water, in some other medium.

5. Lemonade stand.

6. Simple linear programing problem, perhaps after systems of linear equations.

7. An automotive plant produces one kind of automobile and one kind of truck. Each car uses $1\frac{1}{2}$ tons of steel, and each truck 3 tons. Each car when sold brings $300 profit, and each truck $400. The total number of vehicles that the plant can produce in a year is half a million. The total amount of steel available to the plant is 975,000 tons. How many cars and how many trucks should be scheduled for the year's production to maximize the total profit?

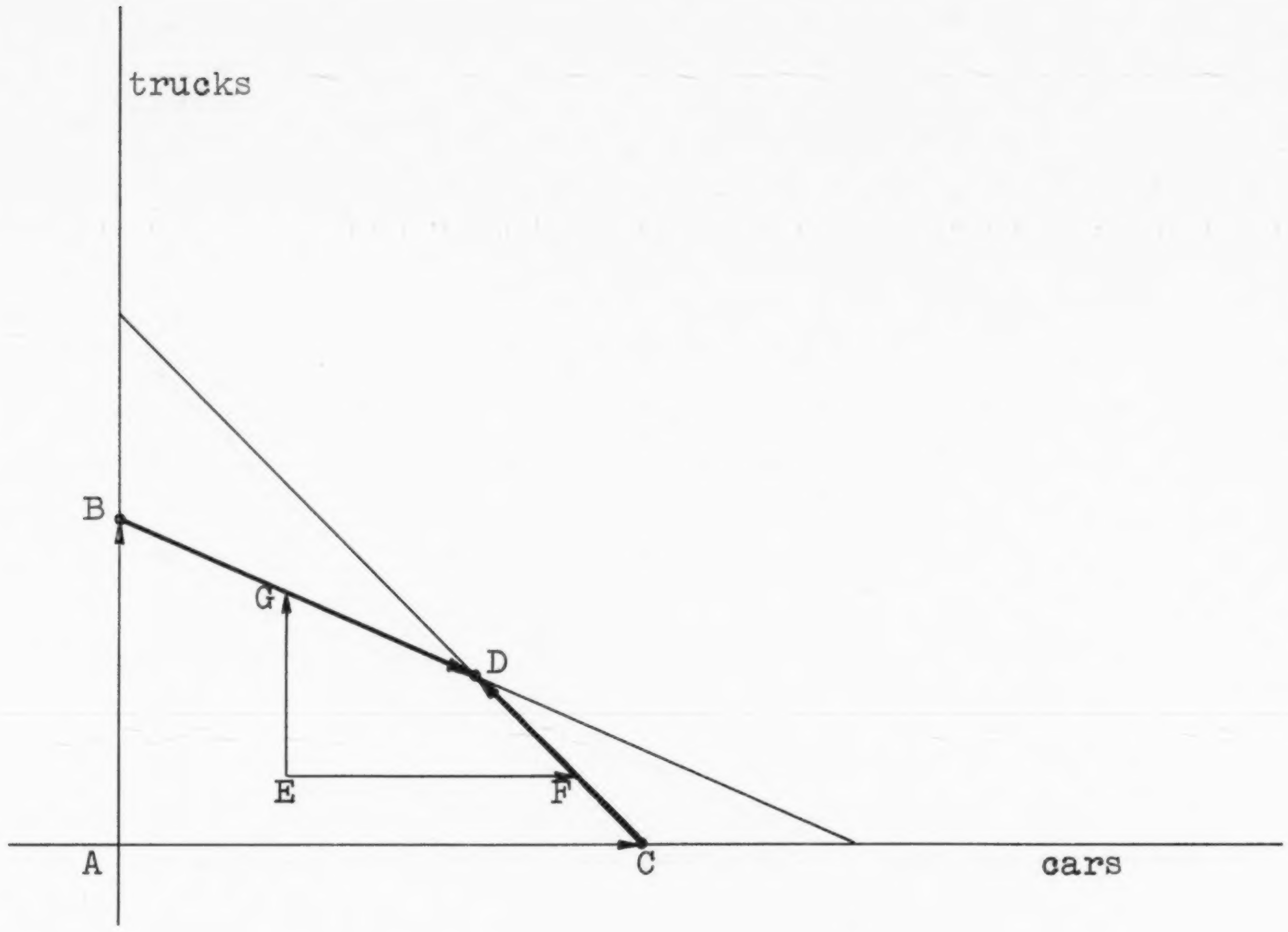
Instructions to the teacher:

a. Draw graphs of tons of steel used in the production of n cars, then n trucks. Note graphs are dots on line segments terminating when n = 0 and n = 500,000 (intersection of straight line segment and integers).

b. Draw graph (lattice of ordered number pairs of cars, trucks) of permitted numbers of cars and trucks satisfying vehicle capacity restriction (constraint).

c. Draw graph of number pairs of permitted production satisfying steel capacity constraint.

d. Draw intersection of the two sets of above lattice points. This set of points is a set satisfying both contraints (as well as non-negativity of production).



e. Note that this set is a convex set, an important notion for optimization problems.

f. Investigate various points on "boundary" of lattice region. Calculate profits as one moves from A to B, A to C, B to D, C to D. Note that profits [300 × (no. of cars) + 400 × (no. of trucks)] increase for all of these directions. Note also increase from E to F and E to G, suggesting nonoptimal character of interior of region.

g. Conclude that optimal "mix" of production is represented by number pair associated with D.

h. Alternatively, draw the lines 300C + 400T = P for various values of P. See what happens as P increases. For what value of P does one of these lines touch ABDC at a simgle point? (Compare with the appendix to Grade 8, Chapter 4, the isoperimetric problem.)

8. There are 5 pickup points--A, B, C, D, and E--for taking students to school in a certain community which is considering the construction of a new school at one of four possible sites--a, b, c, and d. The table of distances is given below.

from to	A	B	C	D	E
a					
b					
c					
d					

The numbers of students to be picked up at A, B, C, D, and E are ___, ___, ___, ___, and ___, respectively. It is desired to choose the site which will result in the minimum total time of travel to and from school by the town's student population. Which site is chosen?

Supply the data according to some real situation in a community.

Let the class propose similar problems: a new housing development and bus service within a certain radius, a town library to be used by townspeople and two high schools, etc.

Let the students collect the data. Can they discover the general pattern common to these problems?

9. The Tunnel. See Studies in Mathematics, Vol. XI, by George Polya, Chapter I, p. 1-4.

This is a problem in applied geometry which could be well used to motivate the study of similar triangles or used after the topic has been studied.

Look also at the remark regarding the generalization of a problem, page 39.

10. Give a student a ruler (preferably a metal tape measure). How many objects are in his classsroom that he can measure? Write a list with the names of the objects and, whenever possible, attach a number to it which represents its measure. Are there any objects that are not measurable? Is there any measure other than the tape which could be used? Now go (or think of going) outdoors. What objects would you find? Which can be measured with the same tape measure?

Professor Polya gave us some ideas on modeling in mathematics instruction. He spent a short period with us, and we also went to his lectures on "Problem Solving."

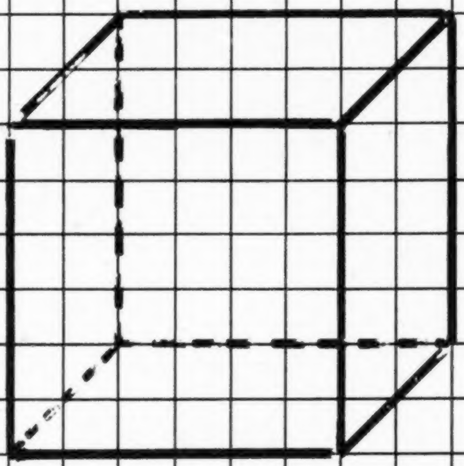
He reminded us that seventh graders are very young and that problems introduced at that age must be appropriate for their level. "They won't believe what they don't understand, and they should not believe what they don't understand." He hoped that in good mathematics education this skeptical attitude is encouraged.

In class a problem led to the solution of a system of three linear equations. He tried to indicate by the use of pieces of cardboard the relation of the planes associated with the equations; he turned to the class, "If you don't know this, believe it!" But he added quickly, "Please believe it half way!"

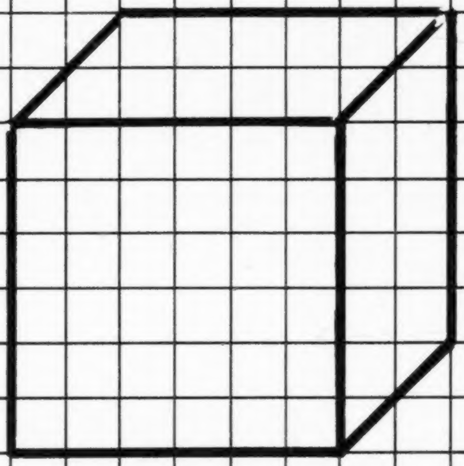
He suggested the use of graph paper in making models of solids. The box (you don't have to use a word like parallelepiped, which is too hard to spell and to pronounce) is the basic principle for drawing many objects around us. It should be used by teachers also so that their drawings on the board would improve.

The following diagrams illustrate Professor Polya's ideas. We think that problems of the following type could be devised:

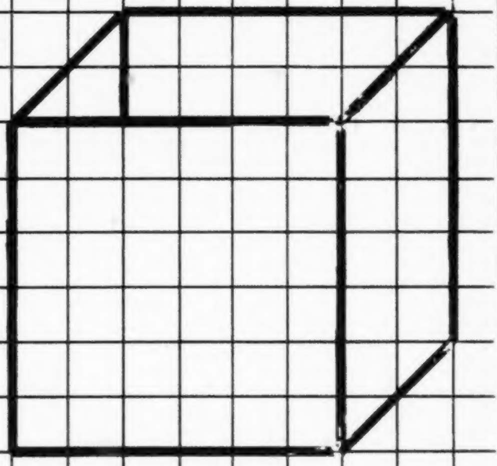
1. Build a toy chest for a little brother (or sister);
2. Build a doghouse.
3. How much material is needed, etc.?



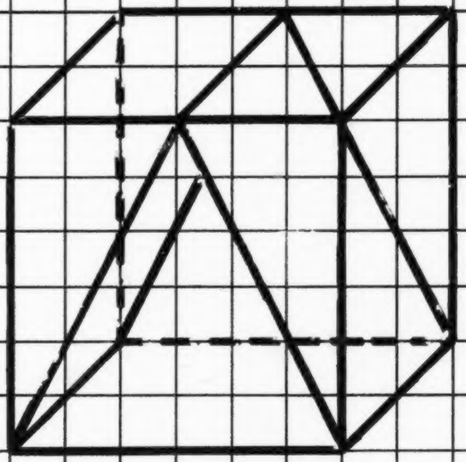
Transparent Material



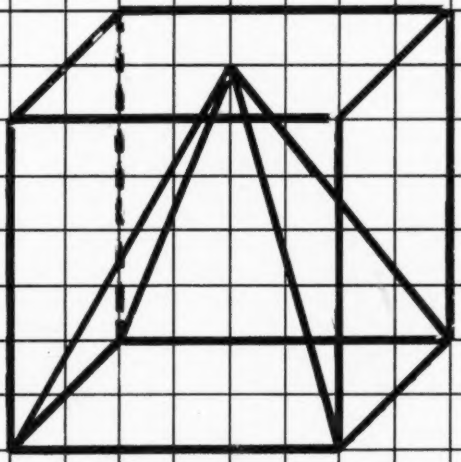
Card Board



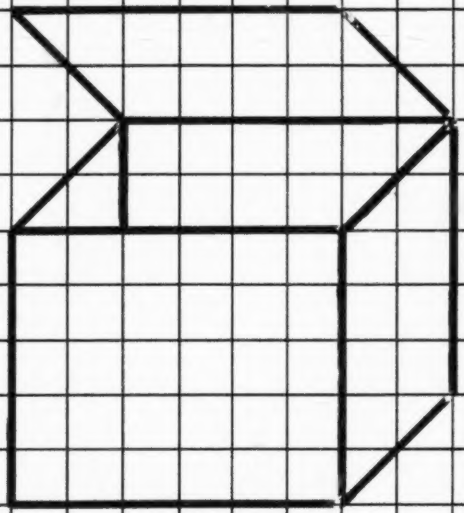
Open Box



Tent or Roof

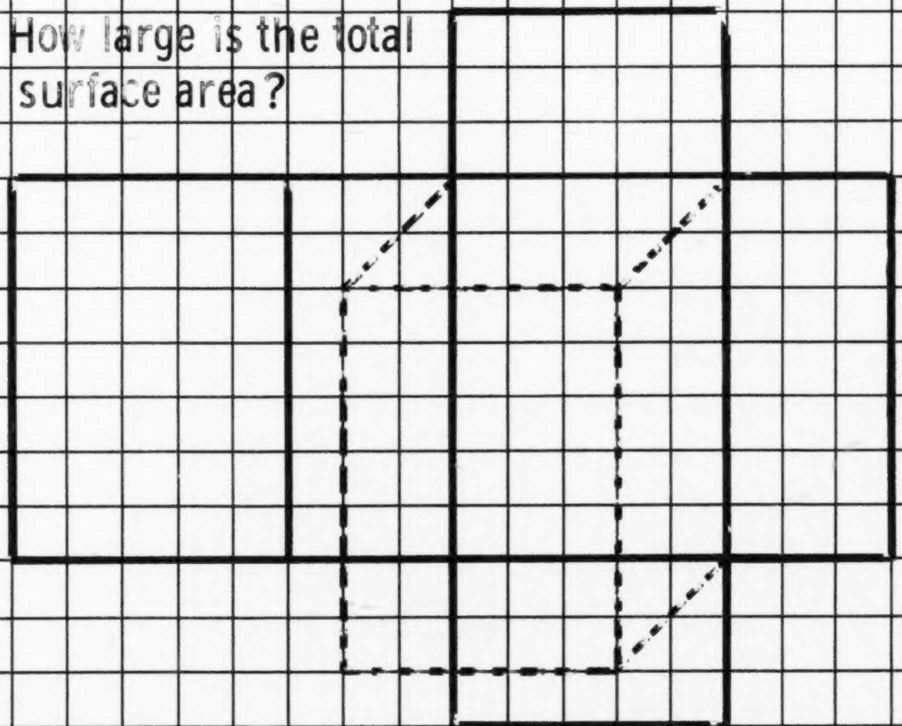


Pyramid or Steeple

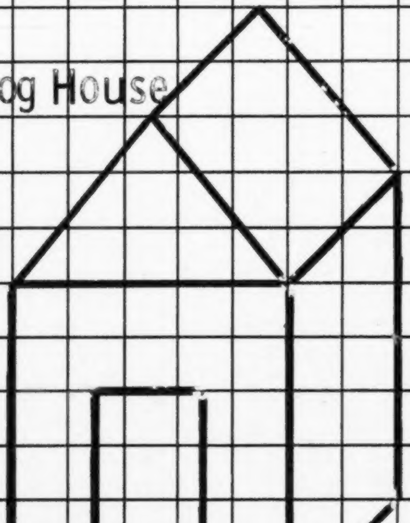


Toy Chest

How large is the total surface area?



Dog House



APPENDIX D

Excerpts and References about Modeling
Passed on with the Compliments of the Modeling Committee

A. Excerpt from the GCMC Report of CUPM on a Course in Applied Mathematics

On Applied Mathematics

At the heart of applied mathematics is the process of model building. What you need to do is to take a situation in another field--it doesn't matter whether this is engineering or physics or economics or biology or what have you--which you would like to understand better and to invent a mathematical model that will (hopefully) help you to understand that situation. You then proceed to analyze this mathematical model, including particular numerical examples if they are relevant, and finally see what you have learned through the mathematical model about the original "physical" situation. Now a course in applied mathematics could be organized around either the field of human endeavor to which mathematics is being applied or around the mathematical discipline being used in analyzing the model. ("Theoretical mechanics" and "methods of mathematical physics" are examples of each of these possibilities.) Our purpose is to organize instead around the process of model building itself. A sequence of situations from various fields of applications could be chosen, for instance, to illustrate each of the following aspects of model building.

1. A mathematical model of, say, a situation in physics must be complicated enough so that it honestly represents the real world without omitting any essential features of the physical situation and yet be simple enough so that you have a fighting chance to do something with it mathematically. Typically these two don't meet at first try, and it is an exciting struggle to obtain a sufficiently simple mathematical model without losing the essence of the problem.

2. When you have made a mathematical model, you have to consider all its consequences, those that you like because they agree with your

physical intuition about the problem as well as those whose physical implications come as a real shock. This frequently leads to a refinement of the model as well as to new problems that need to be formulated and analyzed.

3. The analysis of the first mathematical formulation of, say, an engineering situation may reveal that the engineer doesn't really know what it is he wants to understand, to build, or optimize. The mathematical model serves to focus on the question that actually should be asked: What are we trying to optimize, for instance, or when does the engineer wish to consider two mathematical solutions equivalent? The attempt to build a satisfactory mathematical model forces the right question about the original situation to come to the surface.

4. A model is always an approximation to reality and should therefore be stable with respect to perturbations in the less certain of its mathematical assumptions. If such changes in the assumptions cause a major upset of the mathematical conclusions, then the conclusions may be physically suspect, for we often cannot be completely sure of the precision of our assumptions. Attempts to obtain stable rather than unstable mathematical models are a very interesting aspect of model building.

B. The Goals from R. C. Buck's Article "Goals for Mathematics Instruction" Which are of Particular Importance in Modeling

Goal 1: To provide understanding of the interaction between mathematics and reality.

Goal 2: To convey the fact that mathematics, like everything else, is built upon intuitive understandings and agreed conventions and that these are not eternally fixed.

Goal 3: To demonstrate that mathematics is a human activity and that its history is marked by inventions, discoveries, guesses both good and bad, and that the frontier of its growth is covered by interesting unanswered questions.

Goal 6: To show that complex things are sometimes simple and simple things are sometimes complex and that in mathematics, as well as in other fields, it pays to subject a familiar thing to detailed study and to study something which seems hopelessly intricate.

C. Digest of R. S. Burrington's article "On the Nature of Applied Mathematics," American Mathematical Monthly, (April 1949) 56:221-242.

D. A General Summarization of Modeling

When considering problems that are concerned with applying mathematics to situation in the real world, one is often confronted with the issues in a complex environment full of distraction. It remains to develop a well-organized structure so that the essentials of the problem can be viewed with less confusion. The delicacy of such a task lies in the following:

1. Removal from the original setting of only the barest features of the problem. This requires due examination of the original setting to gain direction in determining that which is fundamental. The result of such an effort is a simplified, idealized concrete or physical model of the original problem.

2. This idealized model is to be made the subject of mathematical investigation by direct translation to mathematical terms, i.e., an isomorphism. Essentially this translation is a mathematical model of the idealized model of the original problem.

3. Through manipulative computation a solution is obtained for the mathematical model.

4. The solution is interpreted in terms of the idealized model.

5. Finally, the solution is interpreted in terms of the original problem.

The validity of the results depends upon the extent to which the models include all of the known pertinent facts.

CONTENTS OF GRADE 7

Chapter 1: The Structure of Space

Section 1: The Structuring of Space in Terms of Point, Line, Plane
- 2: Incidence Properties
- 3: Separation Properties
- 4: Convexity
- 5: Orientation on a Line or on Parallel Lines
- 6: Orientation in a Plane

Chapter 2: Graphs and Functions, Variables

Section 1: Coordinates
- 1.1 Point Plotting
- 1.2 Translation, Reflection, and Symmetry
- 1.3 Add. of Pos. Rationals, Mult. by Pos. Integers

Section 2: Function
- 2.1 Illustrations
- 2.2 Observation of Function as Ordered Pairs

Section 3: Graphs of Functions
- 3.1 Review of Graphs of Section 1.3 as Graphs of Functions
- 3.2 Review of Graphs of Other Relations
- 3.3 Coin Tossing Experiment

Chapter 3:

"Positive Version" (Dean)

The Positive Rationals

Section 1: Sentences and Their Solution Sets.

$\frac{a}{b}$ as solution for $bx = a$

Alternate Version (Corcoran)

The Set of Rationals

Section 1: The Opposite Function
- 1.1 Opp: $x \rightarrow -x$
- 1.2 $-(-x) = x$

Section 2: Arithmetic Operations from the Point of View of Equations

Section 3: Order for the Positive Rationals

Section 4: Decimals
- 4.1 Expanded Notation, Positive Exponents
- 4.2 Extension to Negative Exponents
- 4.3 Scientific Notation
- 4.4 Repeating Decimals

Section 5: Percent (as Function)
- 5.1 $\% : x \rightarrow \frac{x}{100}$

Section 2: Properties of Rational Numbers
- 2.1 Assume that Operations of Addition and Multiplication Exist such that Old Properties (Closure, Commutativity, etc.) Hold
- 2.2 Attention to Existence of Additive Inverse

Section 3: Absolute Value Function
- 3.1 Definition
- 3.2 Distance from 0 on the Number Line

Section 4: Addition and Multiplication of Rationals
- 4.1 Review Addition of Integers
- 4.2 Extend to Addition of Rationals
- 4.3 Multiplication of Rationals

Section 5: Division of Rationals
- 5.1 Definition
- 5.2 Prove $\frac{a}{a} = 1$

Section 6: Subtraction of Rationals
- 6.1 Review Subtraction of Integers

6.2 Practice Subtraction
6.3 Subtraction as Distance on the Number Line

Chapter $3\frac{1}{2}$: Solution of Mathematical Sentences

Section 1: Decimal Names for Rationals
    1.1 Expanded Notation, Positive Exponents
    1.2 Extension to Negative Exponents
    1.3 Scientific Notation
    1.4 Repeating Decimals

Section 2: Ordering the Rationals
    2.1 Basic Order Properties
    2.2 Add. and Mult. Properties of Order
    2.3 Order of Rationas on the Number Line
    2.4 Density of Rationals

Section 3: Introducing Percent
    3.1 As a Function, $\% : x \rightarrow \frac{x}{100}$
    3.2 Practice Computations

Section 4: Solutions of Equations and Inequalities
    4.1 Restatement of Properties of Equality and Order
    4.2 Solutions of Equations and Inequalities of Forms: $x + a = b$, $ax = b$, $ax + b = c$, $ax + bx = c$

Chapter 4: Congruence (Replication of Figures)

Section 1: Congruence of Segments; of Angles
- 2: Division of Segments and Angles into Two Congruent Parts
- 3: Addition Property for Segments
- 4: Subtraction Property for Segments
- 5: Addition and Subtraction Property for Angles
- 6: Vertical Angles
- 7: The Concept of Congruence
- 8: Congruence of a Figure with Itself
- 9: Congruence of Triangles
- 10: The SSS Congruence Property
- 11: The SAS Congruence Property
- 12: The ASA Congruence Property
- 13: Motions by Means of a Coordinate System
  - 13.1 Sliding (Translation)
  - 13.2 Turning (Rotation)
  - 13.3 Flipping (Reflection)

Chapter 5: Measure

Section 1: Linear Units of Measurement
- 1.1 Linear Units of Measurement
- 1.2 Applications of Linear Units
- 1.3 Linear Measure and Circles

Section 2: Angular and Arc Measure
- 2.1 Angle Measure
- 2.2 Sum of Measures of Angles of a Triangle
- 2.3 Arc Measure and Central Angles
- 2.4 Triangle Inscribed in a Semi-Circle
- 2.5 Angles Inscribed in Circles

Section 3: The Pythagorean Property and Applications

Section 4: Equivalence of Polygonal Regions
  4.1 Equivalent Region Building
  4.2 Decomposing Regular Polygons
  4.3 Forming Rectangular Regions

Section 5: "Greater Than" (in a Geometric Sense) for Segments, Angles, Planar Regions, Spatial Regions

Chapter 6: Ratio and Similarity

Section 1: Magnification and Contraction
  1.1 Informal Illustrations
  1.2 Introduction on Coordinate System

Section 2: The Concept of Similarity
  2.1 Begin to Refine Relationship of Previous Illustrations
  2.2 Comparison to Congruence

Section 3: Ratio and Proportion
  3.1 Meaning of Ratio; Symbols
  3.2 Proportion as Equality of Ratios

Section 4: Defining Similarity
Section 5: Sufficiency Properties for Triangles
  5.1 Exploratory Work Leading to AA
  5.2 Exploratory Work Leading to SSS
  5.3 Exploratory Work Leading to SAS
  5.4 Corresponding Lines (Altitudes, etc.) in Similar Triangles

Section 6: Similarity Mappings
  6.1 Applications, from 1.1 and Others
  6.2 Local Maps
  6.3 Scale Drawings and Blueprints
  6.4 Tangent Ratio

Possible Addition:
Section 7: Percentage Problems Using Proportions

Chapter 7: Combinatories and Probability

(From SMSG Text on Probability for Junior High, Chapter 1-6)

Section 1: Fair and Unfair Games (Chapter 1)

Section 2: Finding Probabilities (Chapter 2)

Section 3: Counting Outcomes (Chapter 3)

- 3.1 Tree Diagrams
- 3.2 Pascal's Triangle (Without Binomial Theorem)

Section 4: Estimating Probabilities by Observation (Chapter 4)

- 4.1 Organization of Data
- 4.2 Notion of Average and Expectation

Section 5: $P(A \cup B)$ (Chapter 5)

Section 6: $P(A \cap B)$ (Chapter 6)

Chapter 8:

Dean Version	Alternate Version (Corcoran) Graphs of Linear Functions; Variation
Section 1: Review of Negative Rationals as a Set of Numbers; "Opposite" Function and Its Graph	Section 1: Graphs of Functions 1.1 Review Coordinate System and Association of Points with Their Coordinates 1.2 Graphs of Functions with Restricted Domains
Section 2: Multiplication of a Positive Rational by a Negative Rational	Section 2: Slope and Intercepts 2.1 Review Graphs of Linear Functions in First Quadrant 2.2 Experimental Development of Slope 2.3 Develop Notion of Slope in Terms of Difference of x and y Coordinates, and Notion of y-Intercept

	2.4 Slopes of Special Lines and Sets of Lines
Section 3: Graphs of Multiplication by a Positive Rational	Section 3: A Closer Look at Slope 3.1 Special Cases of $f : x \to mx + b$ 3.2 Increasing and Decreasing Functions 3.3 $f : x \to mx$ in Terms of m as a "Multiplier"
Section 4: Multiplication of a Positive by a Negative, and the Distributive Law	Section 4: Variation 4.1 Direct Variation 4.2 Inverse Variation 4.3 Other Kinds, Such as $x \to kx^2$
Section 5: Multiplication by a Negative Rational	Section 5: Solution of Equations Like: $3x + 2 = 5x - 3$ by graphing $f : x \to 3x + 2$ $g : x \to 5x - 3$
Section 6: Addition and Subtraction Revisited	Section 6: Scale Drawings as Functions
Section 7: More on Opposite Function	
Section 8: Absolute Value Function; Graphs	
Section 9: Applications	
Section 10: Graphing $x \to ax + b$; Role of Parameters a and b	
Also, Special Treatment of Multiplication	

Chapter 9: Solutions of Systems of Equations and Inequalities

Section 1: Solving Systems of Equations

- 1.1 Problem Leading to a System of Linear Equations
- 1.2 Writing Equations in "y-Form"
- 1.3 Graphic Solution
- 1.4 Algebraic Solution by "Comparison"
- 1.5 Practice on Solution of Systems

Section 2: Systems Which Do Not Have Unique Solutions

- 2.1 Graph of System when Lines are Parallel
- 2.2 Algebraic Solution of Such a System
- 2.3 Graph when Lines are Coincident
- 2.4 Algebraic Solution of Such a System

Section 3: Graphs of Inequalities

- 3.1 Graph Which is a Half-Plane, as of $x > z$
- 3.2 Graph Which is Union of Half-Plane and Edge, as of $x \geq 2$
- 3.3 Graphs Where Edge of Half-Plane is an Oblique Line through the Origin, as of $y > x$.
- 3.4 Graphs of Inequalities Involving Absolute Value, as of $|x| > 3$, $|x| < 3$, etc.

Section 4: Systems of Inequalities

- 4.1 Graph which Is Intersection of Half-Planes, as of System

$$\begin{cases} x < 2 \\ y > -2 \end{cases}$$

- 4.2 Graphs of Other Systems, including:

$$\begin{cases} y < x \\ x > -2 \end{cases} \quad \text{and} \quad \begin{cases} x < 5 \\ -3 \leq y \leq 5 \end{cases}$$

Chapter 10: Decimals; Square Roots; Real Number Line

Section 1: Motivation

- 1.1 Recall of Familiar Sets of Numbers
- 1.2 Review of Pythagorean Theorem; Arithmetic Interpretation

Section 2: Numbers Which Are Not Rational

- Section 3: Names of Rational Numbers
    - 3.1 $\frac{a}{b}$ as a Repeating or Terminating Decimal
    - 3.2 How Tell for $\frac{a}{b}$ Whether Its Repeating Decimal Will Terminate?
    - 3.3 Find $\frac{a}{b}$ Name for a Repeating Decimal
- Section 4: Irrational Numbers
    - 4.1 Infinite Decimals which Do Not Repeat
    - 4.2 Iteration Method for $\sqrt{5}$ (Flow Chart?)
- Section 5: Real Number Line
    - 5.1 Location of Rational Points
    - 5.2 Location of Irrational Points
    - 5.3 Completeness of Real Number Line
- Section 6: Properties of Real Number System Emphasis on Density

Chapter 11: Parallelism

- Section 1: Parallel One-Dimensional Objects
    - 1.1 Definition of Parallel Lines
    - 1.2 Skew Lines
    - 1.3 Extension to Rays and Segments
    - 1.4 Network ("Grid") of Equidistant Parallel Lines
- Section 2: Parallel Two-Dimensional Objects
    - 2.1 Plane Parallel to Plane
    - 2.2 No "Skew" Planes
    - 2.3 Line Parallel to Plane
    - 2.4 Extension of Concept by Means of "Carrying" Lines and Planes
    - 2.5 Two Parallel Lines Determine a Plane
    - 2.6 Equations of Lines Parallel to Axes; Inequalities for Strips and 2 - Space Intervals.
- Section 3: Transversals
    - 3.1 Review Definitions
    - 3.2 Define Transversal Lines
    - 3.3 Define Transversal Planes

3.4 Define Dihedral Angles
3.5 Corresponding Angles, Alternate Interior Angles, in 2- and 3- Space
3.6 Parallel Property
3.7 Properties of "Parallel $\Longleftrightarrow$ Congruent Angles"
3.8 Construction of a Line Parallel to a Line through a Fixed Point
3.9 Define Parallelogram, Rhombus, Trapezoid
3.10 Proofs of Theorems about Quadrilaterals
3.11 Constructions
3.12 Segment Parallel to Side of a Triangle; Ratios

Section 4: Transversals to Three or More Lines and Planes
4.1 Three or More Parallel Lines and Transversal Lines
4.2 Three Parallel Lines and Transversal Planes
4.3 Three Parallel Planes and Transversal Lines; Also Transversal Planes
4.4 Intuitive Understanding of Segments Cut Off by and on Transversals
4.5 Median of Trapezoid; Relation to Diagonals
4.6 Nets of Parallels and Coordinate Systems

GRADE 7 - CHAPTER 1

THE STRUCTURE OF SPACE - NONMETRICAL PROPERTIES

Basic Theme:

Most of our knowledge (certainly our scientific knowledge) refers to or is set in the framework of physical space. When a child begins to crawl he discovers geometrical properties of space by pressing against walls, by patiently putting things in cupboards and just as patiently taking them out, by finding paths which take him back to where he started. This knowledge, gained concretely and intuitively over the years, is formulated and structured traditionally in tenth grade geometry.

In this chapter we are studying the most basic threads of this knowledge which are involved in our conception of space. We do not study space as a void, but as "filled" with figures -- some bounded, some unbounded. The particular way in which we do this "filling" or "structuring" or "modeling" determines the nature of our geometric theory. We do it by conceiving (or imagining) the basic figures to be linear: points, lines, planes. These seem the simplest and most natural ones to choose -- they are suggested by familiar objects of experience, dots on paper, fence poles or stakes, taut cords, lines of sight, etc., etc. These concepts certainly can be realized in the physical world to a very high degree of approximation and form what is possibly the most important and useful mathematical model the human race has developed.

In later chapters, we study other, more subtle, aspects of how we conceive (or model) space: congruence -- the idea that figures can be copied freely anywhere in space; measure -- the application of real numbers to compare figures in space and specify their sizes; similarity -- the idea that figures can be "blown up" (or shrunk) uniformly anywhere in space.

One very important outcome of our study is the idea of a coordinate system -- the idea that points in space can be given "addresses", that they can be labelled in a systematic way by specifying certain real numbers which tell us where the points are. This plays a double role: (1) it enables us to study points (and figures also) by means of their addresses, the numbers that label and locate the points; (2) it enables us to take numbers and plot them to find the corresponding points and so to picture and study numbers and numerical relations by means of our model of space.

Purpose:

To review the nonmetrical ("qualitative") properties of space and the associated "linear" figures (point, line, plane, segment, ...) and to consolidate and extend this knowledge as a model of physical space.

Background:

Nonmetrical geometry was begun in Grade 4, Part I, Chapter 5; the ideas have been used in Grades 5 and 6 as needed.

1. The structuring of space in terms of point, line, plane.
   (a) What physical objects suggest the ideas of point, line, plane?

   (b) How do we conceive points, line, planes? How can one interpret point, line, plane (say in classroom)? Do our interpretations correspond exactly to our concepts?

   (c) How are the idea of point, line, plane interpreted by people who are mathematics practically? E.g., scientists, engineers, surveyors, map makers, carpenters, people who lay foundations for buildings, bridge builders.

   Some of the material MJHS, Vol. 1, Part I, Chapter 4, pp. 105-112 can be used here, but it should be amplified and supplemented.

2. Incidence Properties: how points, lines, planes are related to each other.

Chapter 4 of MJHS, Vol. 1, Part I, is a good first approximation. Throughout the treatment reintroduce and reinforce the language of incidence: line contains point; plane contains point, line; point is on line, etc.

(a) Determination properties of points, lines, planes, and linearity (flatness) of planes.

Use counterexamples as well as examples in studying a property, e.g., for "two points determine a line" ask whether two points determine a ray or a circle. Cylinders and cones are good counter-examples for the linearity of a plane.

(b) Intersection properties of lines, planes.

(c) Introduction to non-intersection properties of lines, planes: Parallelism of lines, of planes, of lines and planes; skewness of lines.

Example: Find illustrations of parallel lines, parallel planes, etc., in classroom.

Note: Be sure to introduce the verb "intersect" as well as the noun "intersection" as in the phrase -- "two figures intersect".

3. Separation Properties

MJHS, Vol. 1, Part I, 4-6, 7, 8, 10 is a first approximation -- some tightening is indicated below.

(a) Betweenness of points; the definition of segment (MJHS, Vol. 1, Part I, 4-6).

(b) Idea of ray, half-line.

Review concept of ray (Grade 4, Part I, Chapter 5) and notation.
Give exercises in recognizing rays and specifying them by symbols.
Introduce idea of opposite rays.

Define half-line as ray with endpoint deleted. (Half-line AB is $\overrightarrow{AB}$ with A deleted.) Observe (lightly) that a half-line has an opposite half-line just as a ray has an opposite ray.

A Problem of Terminology: Ray and half-line are examples of "semi-infinite intervals": the first closed, the second open. The term "ray" has been pre-emptied in elementary mathematics for the closed type. In many ways the open type is more useful (e.g., in separation) and more fundamental, particularly because it is easier to adjoin an endpoint (apply set union) than to delete it. But this very useful type of open semi-infinite interval is stuck with the long name half-line. A simpler terminology might be ray (or half-line) for the open type and closed ray (or closed half-line) for the closed type.

(c) Separation of line by point into two half-lines.

Property: Suppose points A, B, C are on line ℓ and B is between A and C. Then B separates ℓ into half-line BA and half-line BC.

Suggestions for class discussion and exercises:

1. Does a point of a segment separate a segment into two figures? If so, can you specify the figures?
2. The same for a ray.
3. The same for a circle; a plane; a sphere.
4. Do two points of a line separate it in some sense? Clarify.
5. The same for a segment.
6. The same for a circle; a plane; a sphere.
7. Into how many pieces is a line separated by 1 point, 2 points, 3 points, 7 points, n points? What kind of pieces?
8. The same for a segment; a ray; a circle; a figure eight.

9. Suppose A and B are points of a line ℓ. Can you describe the figures into which A and B separate ℓ? The same for a segment.

10. The same where three points A, B, C are given.

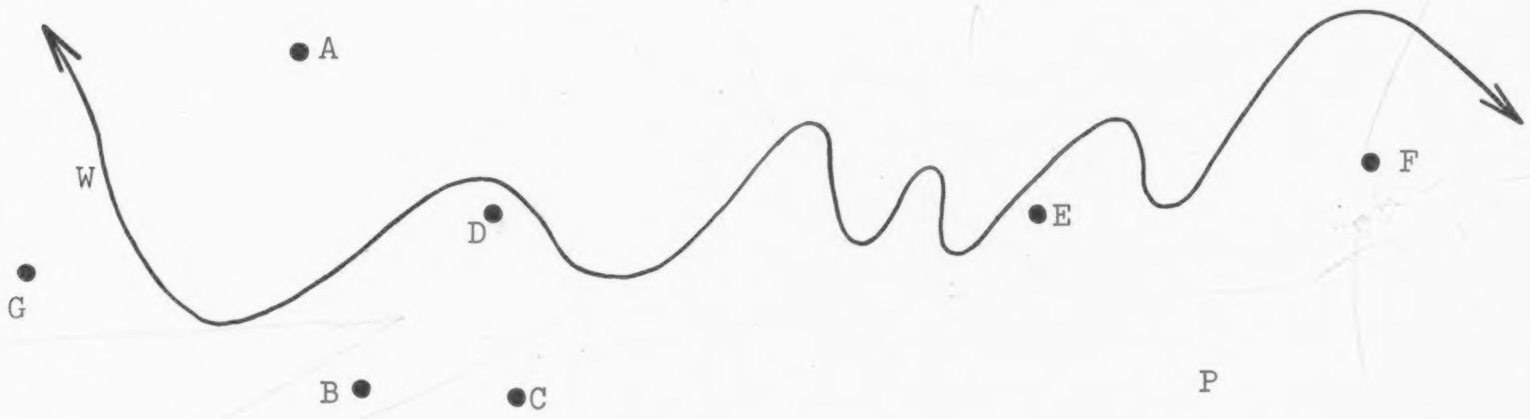
11. Make some separation problems of your own.

(d) The concept of half-plane.

We begin by trying to give some indication of the nature of a half-plane before studying the separation of a plane, just as we had the idea of ray (or half-line) before we studied the separation of a line.

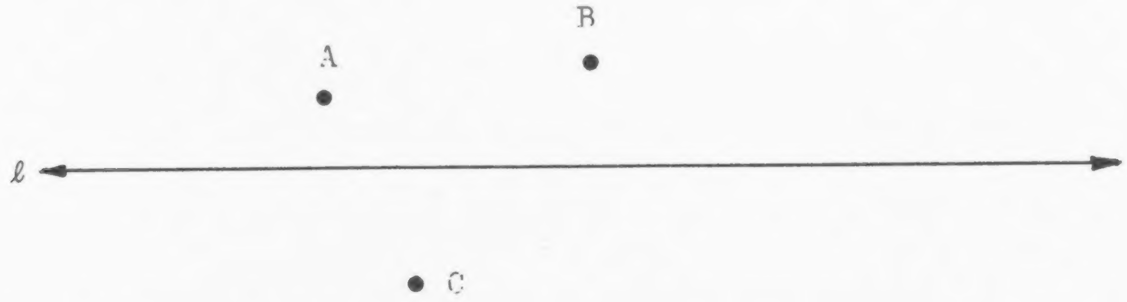
Exploratory Discussion:

(1) Given an endless, wiggley curve w in a plane P, we have:



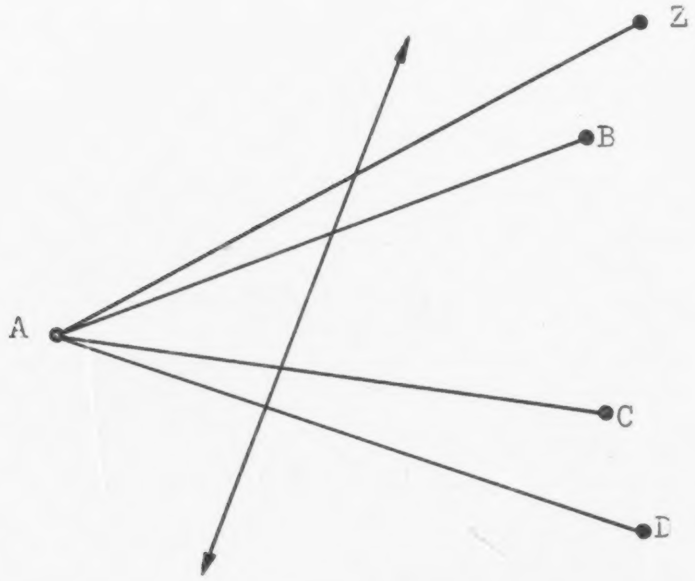
a feeling that w has two sides. A and B appear to be on opposite sides of w, B and C on the same side. Can you make this idea geometrically clear? How could you use geometric ideas that we know to test that A and B are on opposite sides of L ; that B and C are on the same side of L? Will your test work for D and E, A and F, G and C?

(2) Given line ℓ in plane P we feel it has two sides. A and B appear to be on the same line of ℓ, B and C on opposite sides.

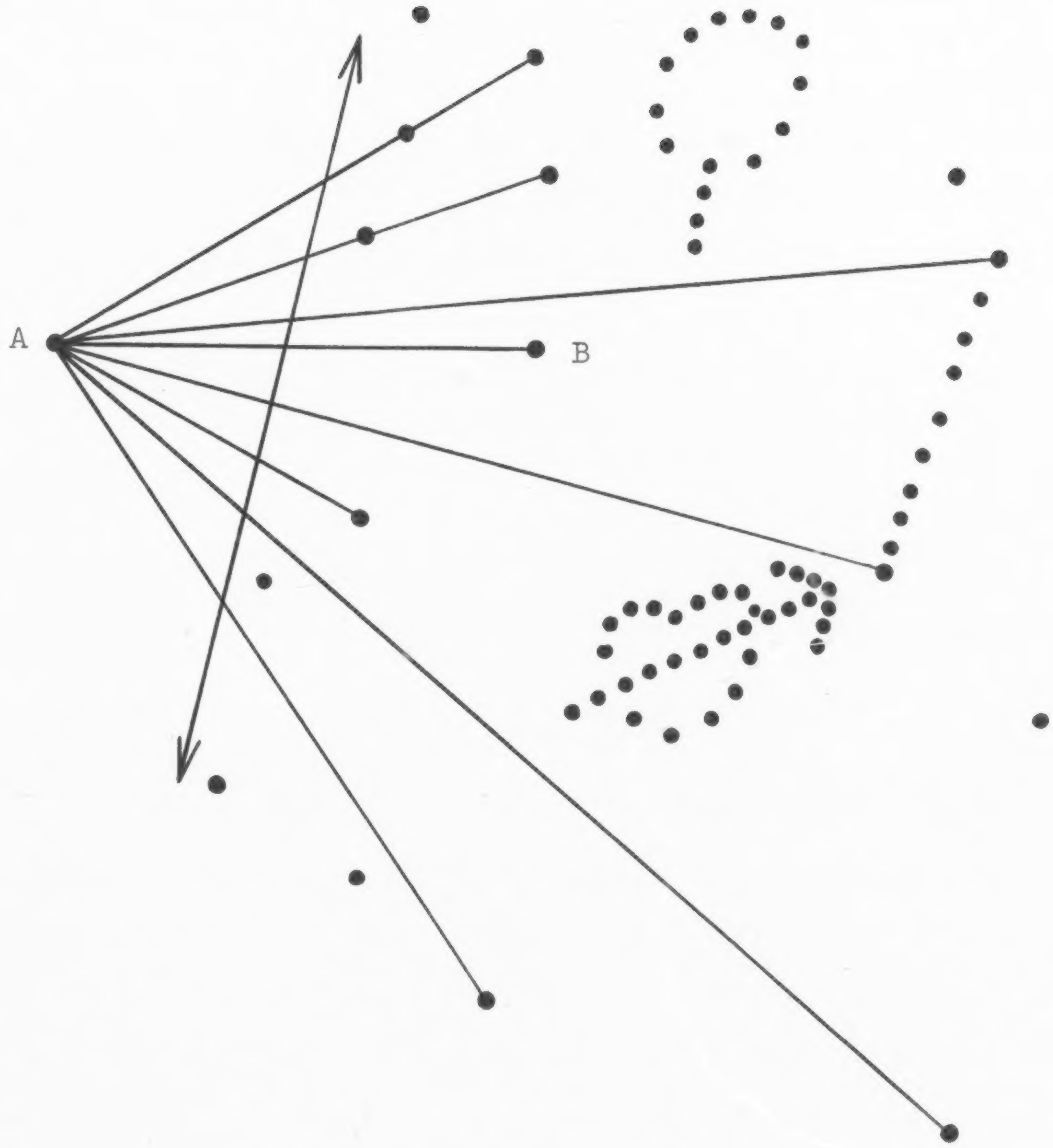


Can you now find a simple test that A and B are on the same side of L and B and C on opposite sides? Choose several additional points and apply your test.

(3) Given line ℓ and point A not on ℓ.



Find a point which is on the opposite side of ℓ from A. Can you find several? How many are there? The set composed of all such points is called a half-plane; ℓ is the edge or boundary of the half-plane. Sometimes we say the half-plane is "opposite" point A.

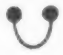


(e) Separation of plane by a line into two half-planes.

Property: Suppose line L is in plane P. Let A, B be points of P such that $\overline{AB}$ intersects ℓ. Then ℓ separates P into two half-planes.

Note: The two half-planes may be described as the half-planes with edge L that are opposite A and B respectively.

Exercises:

(1) Is a plane separated by a ray? A segment? A point? A horseshoe curve like this  ; a "T" ?

(2) Into how many parts do two lines separate a plane? (Or should we ask for Max and Min number of parts?

(3) Into how many parts do three lines separate a plane? (Ask for Max, and Min?)

(4) Same for 4, 5, 6 lines.
See challenge problem at the end of section (g) below.

(5) Does a half-plane contain a segment; a ray; a line?

(6) Does a half-plane contain a half-plane? More than one?

(f) Separation of space by plane into two half-spaces.

Try to treat half-space like half-plane in (d), without necessarily repeating the discussion on the wiggly curve.

Final definition: Given plane P and point A not in P. The set of all points X not in P such that $\overline{AX}$ intersects P is called a *half-plane*. Plane P is its *boundary* or *face*.

Exercises:

(1) Is space separated by a half-plane? A line? A ray? A segment? A point? An open box (conceived as a surface)?

(2) Into how many parts do two planes separate space? (Max, Min?)

(3) Similarly for 3, 4, 5 planes. Compare your results with Exercises (2), (3), (4) of the last section.

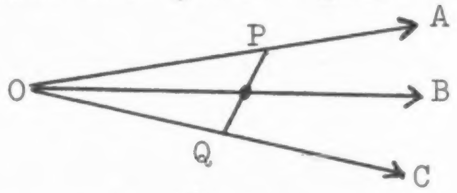
(4) Does a half-space contain a segment, a ray, a line, a triangle, a half-plane, a plane?

(5) Does a half-space contain a half-space? More than one?

(g) Angles and triangles (MJHS, Vol 1, Part I, 4-8); betweenness of rays.

(i) Follow text treatment for angles, interior, exterior, and separation. List the separation theorem as a Geometric Property.

(ii) Introduce betweenness of rays. Exploratory discussion should motivate definition: $\overrightarrow{OB}$ is between $\overrightarrow{OA}$ and $\overrightarrow{OC}$ if $\overrightarrow{OB}$ intersects some segment that joins a point of $\overrightarrow{OA}$ to a point of $\overrightarrow{OC}$.



Exercises I:

(1) Suppose $\overrightarrow{OB}$ is between $\overrightarrow{OA}$ and $\overrightarrow{OC}$ so that $\overrightarrow{OB}$ intersects some segment $\overline{PQ}$, P on $\overrightarrow{OA}$, Q on $\overrightarrow{OB}$ (see figure above). Can you find a segment that joins a point of $\overrightarrow{OA}$ to a point of $\overrightarrow{OC}$ which does not meet $\overrightarrow{OB}$?

(2) Suppose $\overrightarrow{OB}$ is between $\overrightarrow{OA}$ and $\overrightarrow{OC}$, and $\overrightarrow{OX}$ is between $\overrightarrow{OA}$ and $\overrightarrow{OB}$. What betweenness relations for rays follow?

(3) Given $\angle ABC$ draw several rays that are between $\overrightarrow{BA}$ and $\overrightarrow{BC}$. Draw several more. Can you imagine all such rays? What figure do they seem to form (cover)?

(4) Given $\angle ABC$ in plane P. Draw several rays with endpoint A in plane P that are not between $\overrightarrow{BA}$ and $\overrightarrow{BC}$. Exclude $\overrightarrow{BA}$ and $\overrightarrow{BC}$. Draw several more. Can you imagine all of them? What figure do they seem to form?

Exercises II:

(1) Given $\angle ABC$. Can you find a point in its interior? A segment? A ray? A line?

(2) The same for the exterior of $\angle ABC$.

(3) Can you find a ray which separates the interior of $\angle ABC$ into two parts? If so, describe the parts. Can you find another? A third? Compare your answers with your classmates answers.

(4) The same for a segment.

(iii) Sharpen text treatment for triangles, interior, exterior and separation.

Exercises:

(1) Given $\triangle ABC$. Find a point in its interior. Can you find another? Still another? Can you find a segment in the interior of $\triangle ABC$? Another segment? A ray? A line?

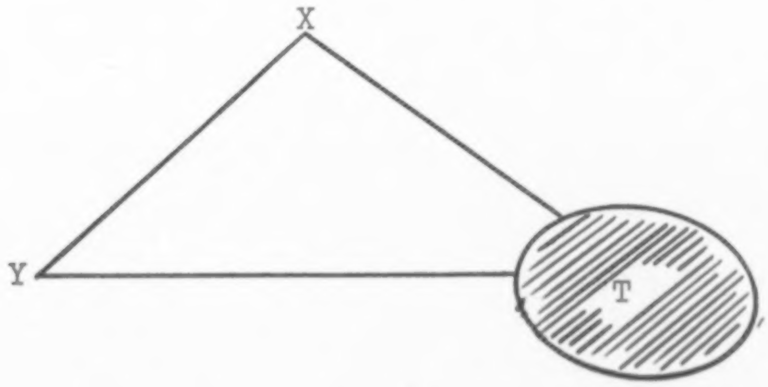
(2) The same for the exterior of $\triangle ABC$.

(3) Can you find a figure that separates the interior of $\triangle ABC$ into two parts? Describe the figure and the parts. Can you find another?

(4) The same for the exterior of $\triangle ABC$.

(5) According to the rules of a game, you are safe only when you're in the interior of a large triangle whose vertices X, R and T are marked by poles with flags, which are joined by ropes from which lanterns hang.

(a) A fog comes up and the corner at T becomes invisible. How can you make sure that you are safe?



(b) The fog thickens and the corner at Y is blacked out. What would you do then?

(c) What would you do if all three corners become invisible?

(d) Suppose X is visible. What is the least information you must know in order to be safe?

(6) Given $\triangle ABC$ draw a segment from A to a point of $\overline{BC}$. Draw several. Try to imagine all of them. What figure do they form?

(7) The same but using B and $\overline{AC}$.

(8) Compare (6) and (7). What geometric fact (facts) is suggested?

(9) Given $\triangle ABC$. Extend its ~~sides~~ to form lines $\overleftrightarrow{AB}$, $\overleftrightarrow{BC}$, $\overleftrightarrow{AC}$. Into how many regions do they separate the plane? Try to describe these regions.

(10) Challenge Problem. Given n lines in a plane in general position -- that is, each line intersects every other line, and no three of the lines meet in a point. Into how many regions do the lines separate the plane? Suggestions:

(a) Try $n = 1, 2, 3, 4$.

(b) Make a table of the function $n \rightarrow R_n$, where R_n is the number of regions for n lines.

(c) Could you get R3 from R2; R4 from R3?

Note: The corresponding problem in space is more difficult and is the basis for Polya's film "Let Us Teach Guessing".

(h) Curves; simple closed curves.

Give some interesting examples of non-planar curves (spirals on a cylinder, the path of a ship (or better, an airplane) that travels constantly on a N.E. course, etc.) as well as planar ones.

(i) Separation of its plane by a planar simple closed curve. Interior, exterior of such a curve; planar regions.

Include in discussion a rich set of examples and counter-examples of separation of surface by a simple closed curve: sphere; torus (surface of doughnut or inner tube); several types of pasted strip: cylinder, Moebius strip, and other twisted strips; possibly surface of pretzel. Maybe surfaces of solid models of letters, e.g., A, B, D, R, etc.

It may be desirable to motivate separation of plane (which may at first seem trivial to the kids) by starting with some of the examples above. Note that the plane separation property is a characteristic property of planes which is not shared by all surfaces.

Make point that human beings don't know innately that space has these separation properties -- they are not trivial, they had to be learned. Consider tiny saucer-like bugs who live on a huge torus. They might not live long enough to discover that their space was not separated by all simple closed curves on it. (Note: The projective plane which is a very respectable mathematical entity, and is closely related to our safe Euclidean world, is neither "separable" nor "orientable".)

Exercises:

Possible Challenge Problem. A child wandered off from his parents in a park that was fenced, but had several gates. Guards at the gates reported he had passed through their gates as follows:

Gate 1 -- 3 times
Gate 2 -- 7 times
Gate 3 -- 5 times
Gate 4 -- 7 times

Where would you look for the child?

(1) Given a circle and its interior. Draw a simple path (curve) which joins two points of the circle and lies wholly in the interior of the circle (except of course for the two points). Does the path separate the interior? Test several paths.

(2) Similarly for a ring bounded by two circles, taking a path that joins two boundary points of the ring.

(3) Generalize to a planar region bounded by several curves.

(4) Given a circular cake, what is the maximum portion of the cake you can get by one continuous cut with a knife through the cake? The minimum?

(5) The same for a cake in the shape of a ring, like an angle cake.

(6) Into how many parts is a sphere cut by a great circle? (Explain the idea of great circle if necessary.)

(7) The same for 2, 3, 4 great circles.

(8) The same for 2, 3, 4, 5 great circles in "general position", that is, no three circles intersect in one point. (Compare similar problems for lines separating a plane in Sections (e), (g).)

(j) Separation of space by a (simple) closed surface (maybe just take one or two examples, e.g., sphere, box, ellipsoid). Interior, exterior of a closed surface; spatial regions. Maybe make this an exploratory exercise.

4. Convexity.

Try the treatment in Geometry Part I, 3-3, selecting exercises from Problem Set 3-3; look over Moise and Downs "Geometry" Problem Set 3-4.

Exercises I:

(1) Is a line a convex set? A ray? A half-line? A segment?

(2) Similarly for a plane, a half-plane, an angle, an angle interior, a triangle interior, a triangle.

(3) Similarly for space, a half-space.

(4) If you take a point away from a line will the resulting figure be convex?

(5) The same for a plane, for space.

(6) The same for a ray, a half-line, a segment.

(7) Is the figure composed of a line and a point not in the line convex? Briefly stated: If we add (or adjoin) a point to a line is the resulting figure convex?

(8) The same for a plane, a ray, a segment, a half-line, a half-plane.

(9) How many different types of convex sets can you find on a line?

(10) The same for a plane; for space.

(11) What can the intersection of two lines be? Two segments; two rays?

(12) What can the intersection of two triangular regions (union of triangle and its interior) be? Will the intersection be convex?

(13) Suppose set A is given. When you take a certain point away from A, the resulting set is convex. Can A be convex? Must A be convex?

Exercises II:

(1) Given two points P and Q.

(a) Can you find a convex set S that contains P and Q?

(b) Can you find a "larger" one (that is, one that contains S) that contains P and Q?

(c) Can you find a "smaller" one (that is, one that is contained in S) that contains P and Q?

(d) Can you find one that is neither "larger" nor "smaller" than S?

(e) Is there a "largest" convex set that contains P and Q (that is, one that contains all the others)? Is there a "least" convex set that contains P and Q?

(f) Is there a least convex set that contains three noncollinear points P, Q, R?

(g) The same for the union of $\overline{AB}$ and point C.

(h) The same for four coplanar points P, Q, R, S.

(i) The same for four noncoplanar points P, Q, R, S.

(j) The same for the union of $\overrightarrow{AB}$ and C; of $\overleftrightarrow{AB}$ and C.

Note that (j) is touchy since the answer depends on the parallel postulate. The problem still seems valuable and can be approached as a limiting form of (g) for larger and larger $\overline{AB}$.

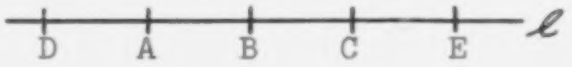
Exercise. Is the union of two convex sets convex? Try simple cases, e.g., triangular regions, segments, convex quadrilateral regions.

Exercise. Can you find a figure whose union with a circular region is convex? Can you find a convex one?

Exercise. Have teacher sign a waiver if this is used. Suppose you're standing in a room and you can shoot anybody in the room with your water pistol. Must the room have a convex shape? Explain.

5. Orientation.

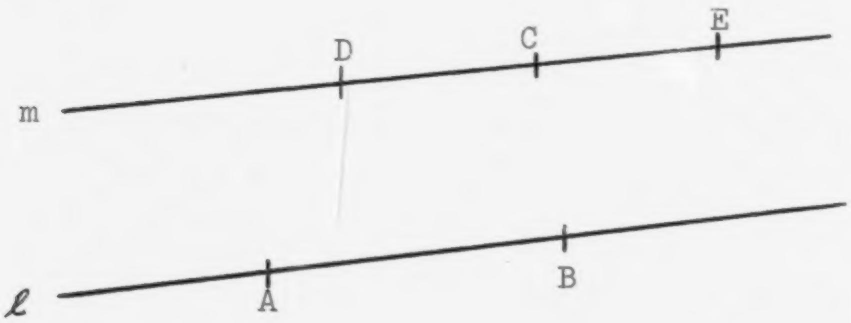
(a) Orientation on a line or on parallel lines.

Given points  on line ℓ

we observe that the "trip" from A to B and the "trip" from C to E are rightward; the "trip" from C to D is leftward. We say the sense from A to B on ℓ is the same as the sense from C to E, but is different from or opposite to the sense from C to D. Briefly, A-B and C-E have the same sense and both have different sense from C-D. (The notion of ordered pair of points (A,B) or directed segment or arrow $\vec{AB}$ are implicit.)

This is probably best approached by discussing a trip, referring to highway markers US 1 North, US 1 South, etc. Give student opportunity to conclude these are just 2 "senses" on the line, that is, we can find two pairs, say A,B and C,D (see figure above), such that every pair of points of the line has the same sense as A-B or the same sense as C-D. Note that A-B and B-A have opposite senses. Maybe use notation A,B instead of A-B?

Extend idea to pairs of points on parallel lines.



Exercises:

(1) Same finger exercises in recognition of senses of pairs in a diagram.

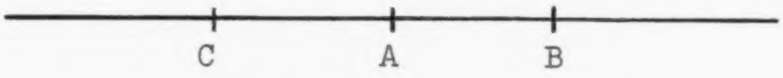
(2) Suppose B is between A and C. What can you conclude about the senses of A-B, B-C, A-C?

(3) Given three points A, B, C on a line. Can you find a condition involving sense of pairs of the points that guarantees B is between A and C?

(4) Suppose A, B, C, D are collinear and B is between A and C; C is between B and D. What can you conclude about senses of pairs of these points? See (2).

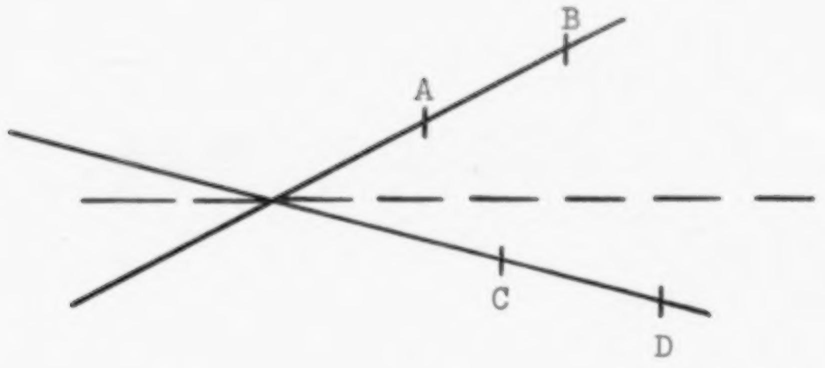
(5) Given four points A, B, C, D on a line. What must you know about senses of pairs of the points in order to conclude that B is between A and C and C is between B and D? See (3).

Discuss rays on a line and bring out that they have a natural orientation, e.g., $\overrightarrow{AB}$ is rightward, $\overrightarrow{BC}$ leftward, etc. Try to get the kids to develop a criterion for two rays on a line having the same sense: $\overrightarrow{AB}$ and $\overrightarrow{CD}$ have the same sense if and only if $\overrightarrow{AB}$ contains $\overrightarrow{CD}$ or $\overrightarrow{CD}$ contains $\overrightarrow{AB}$.



Discuss rays with opposite sense. Relate sense of rays to sense of point pairs. Briefly discuss sense of rays on parallel lines.

Indicate that there is no natural way to define sense of point pairs on two intersecting lines.



We might be tempted to say A-B and C-D have same sense, namely, rightward, but the dotted horizontal line suggests they have opposite sense, namely, A-B upward and C-D downward.

(b) Orientation in a plane.

Try treatment along lines of Prenowitz and Swain "Congruence and Motion in Geometry", D. C. Heath, pp. 31-32, pp. 46-47. Try to use some of the material in Problems 72-76, p. 48, and the theorems, p. 49.

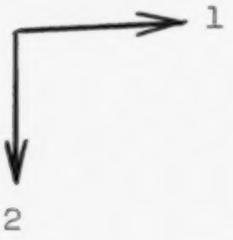
Note: Just as we referred to sense of point pairs, it may be good to speak of sense of pairs of rays $\overrightarrow{OA}$, $\overrightarrow{OB}$ (with a common endpoint). And, of course, sense of "directed segments" or "arrows" corresponds to sense of "directed angles".

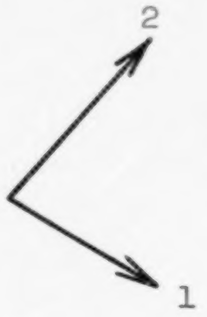
Try to give some exercises relating sense in the plane to compass directions N, E, S, W. For example: If you face north and make a quarter turn in which direction will you face? If you face north and make a "three-quarter" turn and are then facing east, what was the sense of the turn? If you face north and make 13 successive quarter turns, all with the same sense, how will you be facing? This can be modified for a combination of quarter, half and three-quarter turns. If you faced north, closed your eyes, and turned and found yourself facing west, what kind of turn could you have made?

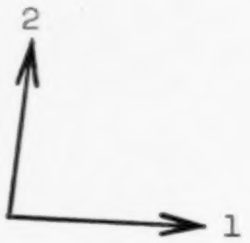
If you face north in front of a mirror, how does your image face? If you then make a quarter turn clockwise, how do you face? How does your image face? What kind of turn did your image make?

Note that you can compare orientation for two parallel planes, just as for parallel lines.

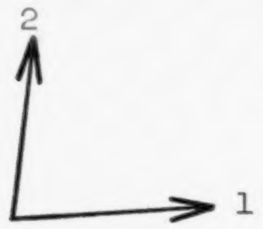
Exploratory discussion. Make many wire figures to indicate right angles, put arrows on ends labeled 1 and 2. Put them on a table and ask kids to try to get them to fit so



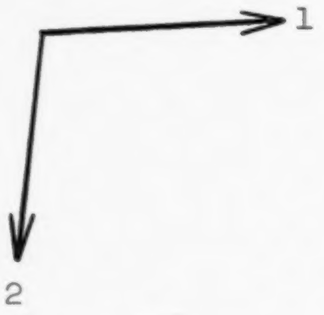




that the "1"-ends coincide by sliding and turning *on the table*. When this is done, how many piles are there? There should be, of course, just 2 -- this indicates that there are just 2 senses in the plane. Observe that

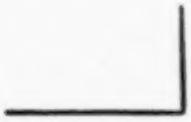


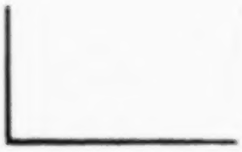
and



cannot be gotten to fit in the way described.

Repeat the procedure using bent wires to represent two unequal segments that meet to form a right angle -- make many such figures which are all congruent. Do *not* label endpoints.





(c) Orientation in Space.

Try to use corkscrews and wood screws to get across idea of right-handed screw -- try to get a left-handed wood screw and drive it into a piece of wood.

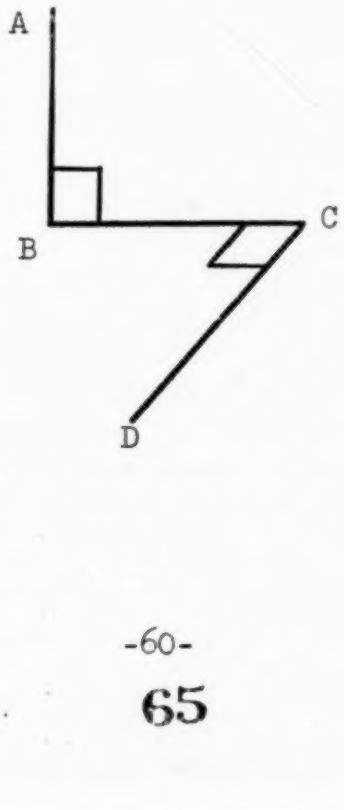
Make comparisons of right and left hands, gloves, and the difference in orientation of an object and its image in a mirror.

Make many wire models of three mutually perpendicular rays, with arrows on endpoints labeled 1, 2, 3 and repeat experiment of (b) for 2 rays.

Compare with figures formed by thumb, index finger and middle finger of right hand and same for left hand.

Make several congruent wire models of 3 mutually segments of different lengths and repeat experiment.

Do same thing using paper clips bent as indicated:



$\overline{AB} \perp BCD$, $\overline{BC} \perp \overline{CD}$.

GRADE 7 - CHAPTER 2

GRAPHS, FUNCTIONS, VARIABLES

(It is appropriate to note that a chapter with this title would be subject to considerable criticism. Hence, a less terrifying title should be sought.)

Background Assumptions:

The student is familiar with the positive rationals together with addition, subtraction, multiplication, and division of positive rationals. Negative whole numbers have been introduced, but not the negative rationals. Coordinate systems in the plane using all four quadrants are known, but with integer coordinates. In Chapter 1 the geometrical concepts of points, lines, planes, etc., have been introduced from the point of view of abstractions of physical situations; hence, a "modeling".

Rationale:

Physical situations are to lead to an introduction of a coordinate system in the plane as a means of indicating that "local" problems often require more than the "global" geometric properties in their solution. Physical situations will also lead to graphs, functions, and graphs of functions. The situations chosen should reflect the basic reason for obtaining functions and their graphs; to wit, that this analysis yields information of a global sort not apparent from local observations. Properties of the function -- as an entity itself -- are discernable and can be used in a predictive fashion. Try to point this out over and over.

Purpose:

We propose that functions be introduced early and used where appropriate without belaboring the concept as a concept, such as happened with sets.

Graphing will permit review of the multiplication of positive rationals. Moreover, the use of letters to represent the coordinates of a point _might_ make the transition to statements $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$ more meaningful.

Section 1. Coordinates:

1.1 A problem should introduce the student to the need for "local" information, beyond geometric concepts, for the analysis of a problem.

Example: An east-west road and a north-south road intersect in front of school. The English teacher lives directly east of school and the librarian lives directly north of school. If both teachers leave home at eight o'clock in their cars and each travels at a constant speed, will they crash at the schoolyard corner?

(Note: Insufficient information to solve the problem has been done on purpose, but this view should not be labored; merely point out that sometimes this does happen. In the meantime, we know that, at least, we need a coordinate system.)

Now turn to graphing proper. In all four quadrants with integer coordinates; with rational coordinates in the first quadrant.

Here the treatment in _Introduction to Secondary School Mathematics_ - Chapter 19, page 301ff should be noted; also Grade 6, Chapter 5 (SMSG).

The emphasis is a pedagogical way to involve the student. Students enjoy some of the "games" of Grade 6, such as, "what ordered pairs give rise to points forming a letter A?" -- or a triangle or a square.

Treat lightly the one-to-one correspondence between points and ordered pairs! Show that the coordinates for a point change as the coordinate axes change -- both translationally and in scale.

Use (a,b) or (c,d) or (x,y) to denote the coordinates of a generic point. (This should be an unsophisticated introduction of variable -- no definition thereof.)

A typical problem:

(1) Given the point whose coordinates are (c,d); plot $(c + 2, d)$, $(c, d + 2)$; $(c - 1, d)$, $(c, d - 1)$; $(\frac{c}{2}, d)$; etc.

1.2 Translation, Reflection, and Symmetry.

(See Grade 6, Part 1; Grade 9, Part II)

(a) Grade reflection about vertical axis $(c,d) \rightarrow (-c,d)$
(This could be done from the point of view of "oppositing" on lines parallel to the number line!)

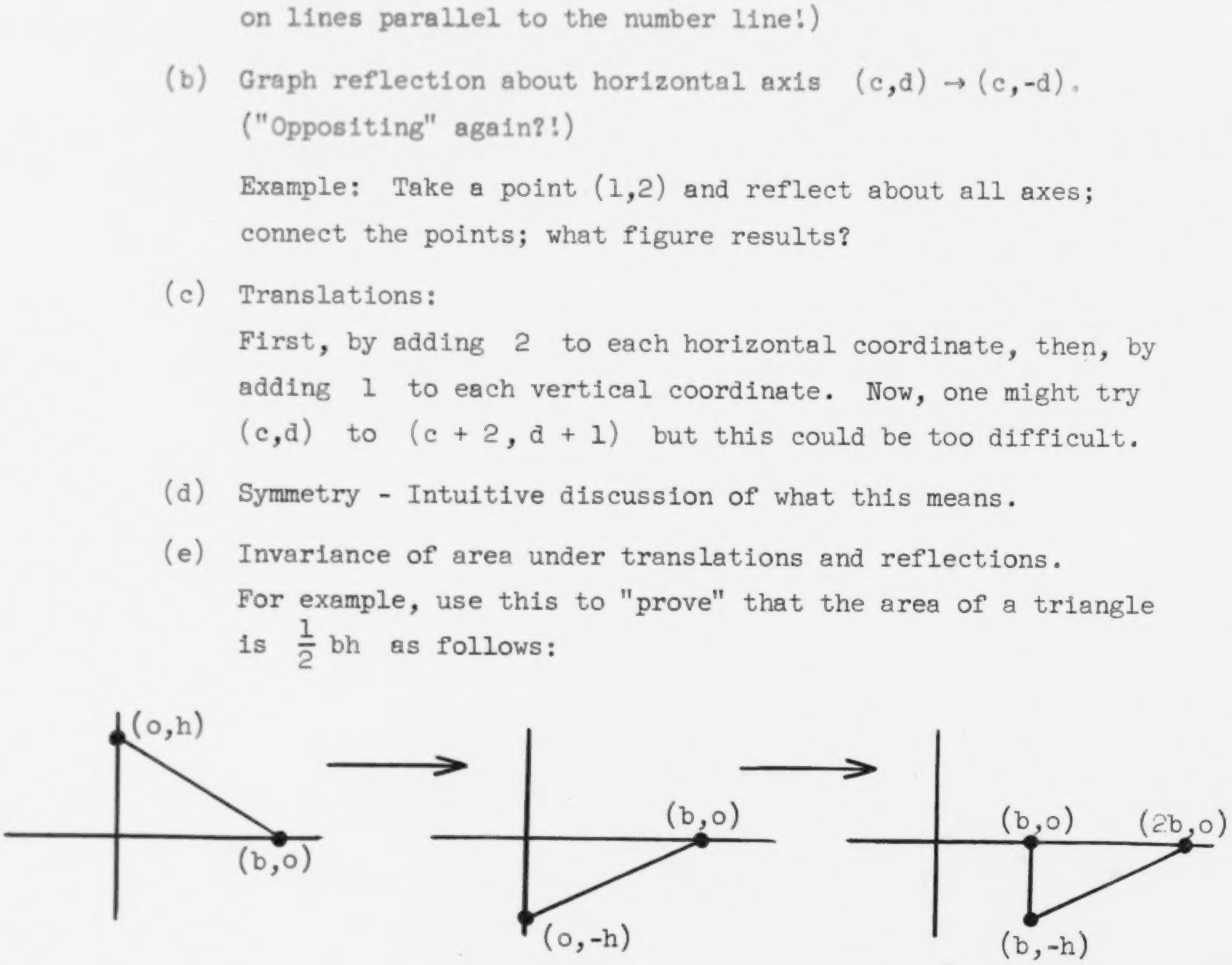
(b) Graph reflection about horizontal axis $(c,d) \rightarrow (c,-d)$.
("Oppositing" again?!)

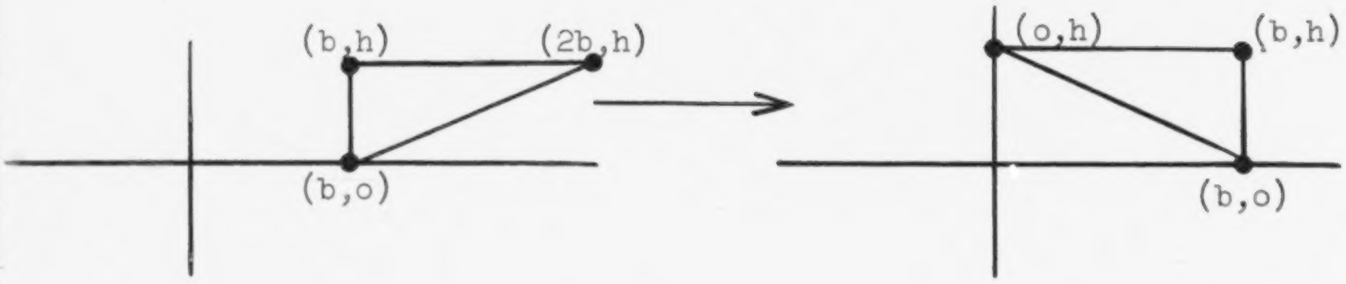
Example: Take a point (1,2) and reflect about all axes; connect the points; what figure results?

(c) Translations:
First, by adding 2 to each horizontal coordinate, then, by adding 1 to each vertical coordinate. Now, one might try (c,d) to $(c + 2, d + 1)$ but this could be too difficult.

(d) Symmetry - Intuitive discussion of what this means.

(e) Invariance of area under translations and reflections.
For example, use this to "prove" that the area of a triangle is $\frac{1}{2}$ bh as follows:





Warning: This section must be tied together with the geometry done either previously or later.

1.3 Graphs which lead to discovery of linear functions.

The section permits considerable practice in reviewing the addition and subtraction of positive rationals, multiplication of positive rationals. Plotting will be largely in the first quadrant, but the student will be encouraged to "discover" what the extension in the 2nd, 3rd, and 4th quadrants might be. This could lead to the student wanting (?) to define $2(-x)$.

Example 1: Plot the points: $(1, 1+2)$, $(\frac{1}{2}, \frac{1}{2}+2)$, $(\frac{2}{3}, \frac{2}{3}+2)$; $(2, 2+2)$, $(\frac{4}{3}, \frac{4}{3}+2)$, $(\frac{1}{16}, \frac{1}{16}+2)$, etc.

Observe linear nature - draw line.

Question: For what a is $(a, \frac{7}{3})$ on this line? Do this graphically also solve $a + 2 = \frac{7}{3}$.

Question: What are the conditions on (c,d) such that (c,d) is on the line?

Example 2: Plot the points: $(2, 2 \cdot 3)$, $(3, 3 \cdot 3)$, $(1, 1 \cdot 3)$, $(\frac{1}{3}, \frac{1}{3} \cdot 3)$, $(\frac{2}{3}, \frac{2}{3} \cdot 3)$, $(\frac{1}{16}, \frac{1}{16} \cdot 3)$.

Observe linear nature - draw line.

Question: For what a is $(a,5)$ on this line? Do graphically: also $a \cdot 3 = 5$.

Write the condition of (c,d) such that (c,d) is on the line.

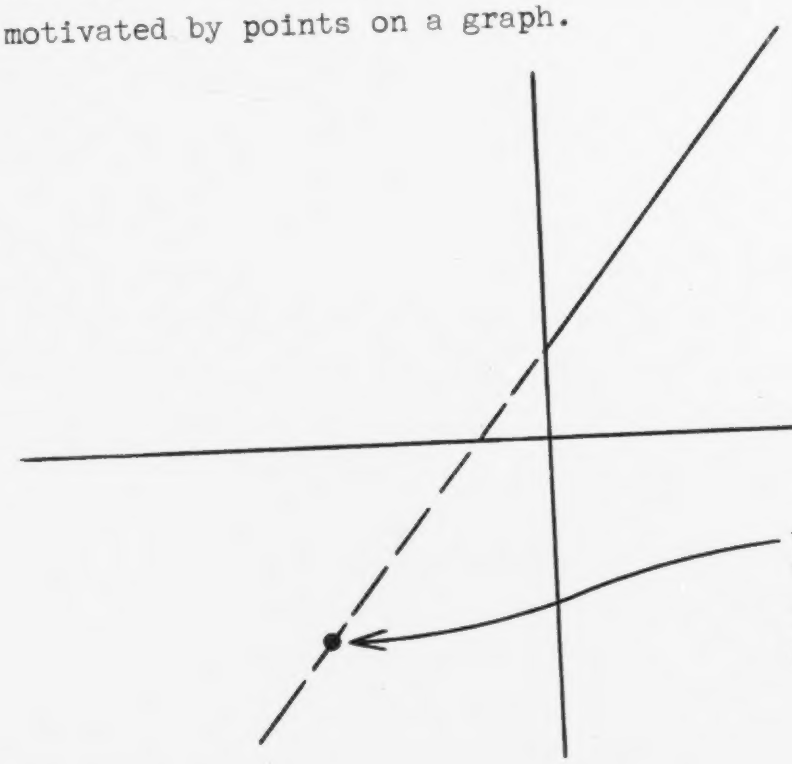
Example 3: Plot the points: $(1, 1 \cdot 3 + 2)$, $(\frac{1}{2}, \frac{1}{2} \cdot 3 + 2)$, etc. Carry out same procedures as above.

Other examples shall be used so that the student has a feeling for $(x, mx + b)$ in the first quadrant for various m and b. Expand to include some negative b, and $m = -1$, although we cannot use other negative m yet.

It is important that the key graphs to be known "cold" are (x,x) and $(x,-x)$ (or $y = x$ and $y = -x$ if this form is preferable). Also the student should know the constant function (x,b) and the vertical lines (b,y).

At this point further talk about the "opposite function" is both possible and desirable. However, the writers may find the shift from a linear to a planar presentation of the idea might be confusing to the student. HANDLE WITH CARE!!

The writers may wish to try at this time a definition of $a(-b)$ motivated by points on a graph.



this point being on the
"natural" extension of a
line "implies"
$a(-b) = -(ab)$.

GOOD LUCK!

Section 2. Function.

Words of Warning: It is essential that "function" be treated somewhat informally and the writers should have in mind the fact that the concept of "function" will contribute preciseness and brevity of language throughout the texts.

Since this idea has not been attempted at this level before, many versions will probably be necessary. The following is one version:

2.1 Illustrations:

These illustrations (examples) are to show that the students' previous experience has often been concerned with an association of objects of one set to objects of another set. Indeed, this notion is so pervasive a notion as to warrant its formulation for more intensive study in itself. The associations will be represented in many ways. We shall want to return as often as possible to our "rationale".

(a) Sons → Fathers - an easy, natural relationship
Father → Son - an easy relationship, but ambiguous for large families!

(b) Persons → Weights - Here is an opportunity to point out that the function as a whole gives information that piece-by-piece data does not. Student might suggest: What is maximum weight? What is average? What is range? How do weights cluster around the average?

(c) Person → Height - Similar comments as in (b) above. How do the two functions correlate?

(d) Plane figure → Area

(i) Just the association - easy example without much structure.

(ii) Restrict domain to regular polygons of side 1. Use cross-ruled paper to estimate area. Here one can study the properties of the graph, showing rate of growth as the number of sides increases. Is it linear?

(iii) Restrict domain to squares of side a.

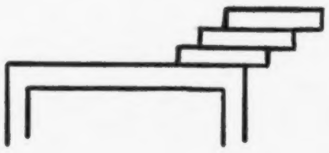
(iv) Back to (i) -- investigate measure properties:

$$M(A \cup B) \leq M(A) + M(B)$$
$$M(A \cup B) = M(A) + M(B) \quad \text{if} \quad A \cap B = \emptyset$$
$$A \leq B \Rightarrow M(A) \leq M(B)$$

(This, of course, must be done with a light intuitive touch, but it emphasizes our rationale.)

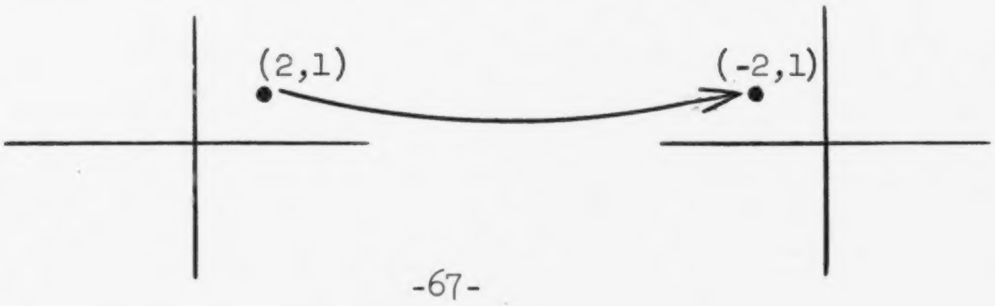
(e) Pulse Rate - See experiment in "Math. through Living Things". While we cannot at this stage compute percents, we can take "first differences" and so construct a new function $D(x) = P(x + 1) - P(x)$. Ask about symmetry of the pulse function P.

(f) Stacking and Overlapping Books (Math. through Science, Part 1)



Again take first differences: should get something like a harmonic series.

(g) Translations and Reflections: Take the ones from Section 1.2. Use pictures to illustrate.



2.2 More careful observation of function as ordered pairs:

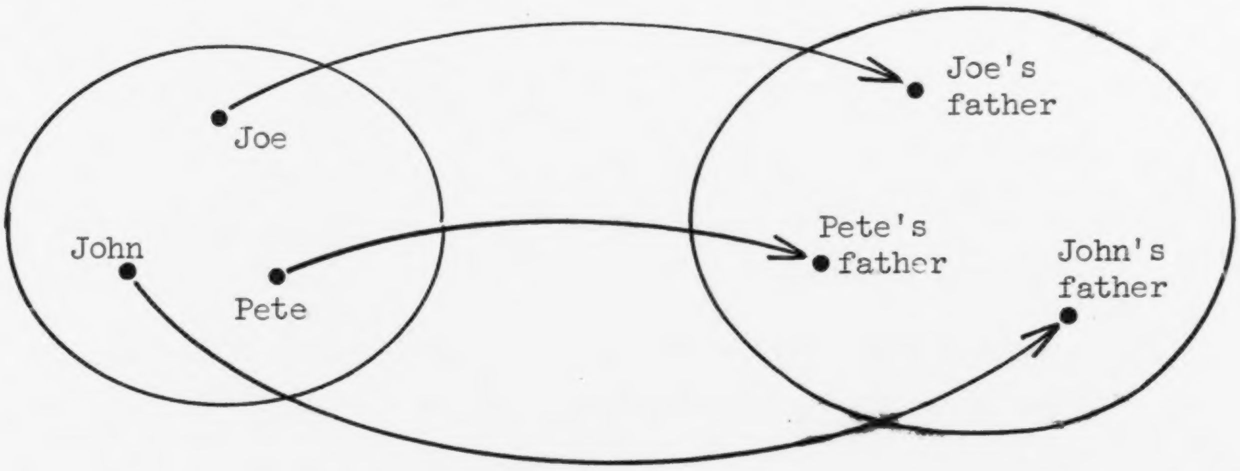
In this discussion the main emphasis is still on "association" as key ingredient of function. However, it should be pointed out that you really don't "know" a function until you know the ordered pairs which constitute the association or function. Thus, we can know the existence of a function (such as the daily noon temperature at the North Pole) without knowing the function. Whether or not a function should be defined to be a set of ordered pairs is still an open question. In any case, we should not make a fetish out of this representation as ordered pairs.

Give counterexamples to show that not every relation is a function: Gradparent → grandchild seems natural.

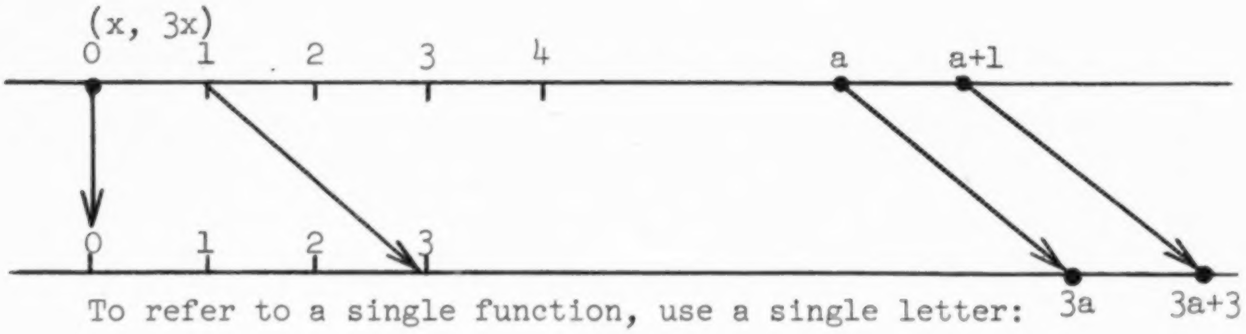
Give a set of ordered pairs and ask the class to guess the rule.

Point out that ordered pairs do not have to be ordered pairs of numbers.

Notations: Show mappings - for example on the son-father association:



Show arrow from one line to another:



To refer to a single function, use a single letter:

f, g, h, A, S, T, etc.

F - for fatherhood: $$F : \begin{cases} \text{Joe} \to \text{Mr. Jones} \\ \text{Pete} \to \text{Mr. Brown} \\ \text{John} \to \text{Mr. Wilson} \end{cases}$$

W - for weight: $$W : \begin{cases} \text{Joe} \to 120 \\ \text{Pete} \to 150 \\ \text{John} \to 110 \end{cases}$$

M - for multiplication: $M = c \to 3c \quad (e > 0)$

r - for reflection about vertical axes:

$$r = (a,b) \to (-a,b)$$

Generically, $f : x \to f(x)$

or $f : x \to y$

(y is determined from x by a rule) and reinforce by examples above.

Section 3: Graphs of Functions

3.1 Review graphs of Section 1.3 - now recognize as functions. Use graphs to solve equations and inequalities. Try $3x + 2 = x + 5$ by graphing $f : x \to 3x + 2$, $g : x \to x + 5$ and getting point of intersection.

Check result.

3.2 Review graphs of

(a) height and weight relations.

Is it a function? Average weights for some height to get a function if necessary. Any conclusion?

(b) Pulse rate:

(c) Area of square of side a.

3.3 Coin tossing:

(a) Have each class member toss a coin 10 times and record (number of heads, number of tails) = x. Now consider the function whose value at x is the number of students whose result is x.

(b) Same data, but consider the function whose value at x is the number of students whose result is less than or equal to x. How is this obtained from (a)?

(c) Re-do (a), normalizing so that value at x is the fraction of the class whose result is x. Re-do (b) in like manner. (The idea behind this problem is to give the students a "distribution" function which can lead to the chapter on Probability.)

GRADE 7 - CHAPTER 3

(The "Positive Version")

THE POSITIVE RATIONALS

This version of Chapter 3 implements the view that there is a lot to learn about positive rationals which has nothing or little to do with their "opposite numbers" and indeed for several chapters (measure, ratio, and similarity, probability) there is little or no need for them. What is needed in these chapters is an early introduction of the notion of a mathematical (open) sentence and a solution for the same. In addition some review is needed of the algorithms for the addition and multiplication and decimal notation for positive rationals. Other new topics are order, density, inequalities in two dimensions, and percent.

One other advantage is that this work with positive rationals as well as the succeeding mathematics gives the student a great deal of added mathematical maturity and readiness for the operations with negative numbers.

Background:

From Chapter 2: Graphing and the whole set of rationals. From Grade 6: Arithmetic operations on + and x. Also the concept of rational numbers as quotients of integers (Grade 6, Part II, p. 369) and a minimal introduction of solving $a \times n = b$ (see below).

Section 1. Sentences and their solution sets.

A nice introductory treatment occurs in ISSM, Vol. II, Part 2, p. 279 which could be taken over if it is modified to work also with $3x = 12$ as well as $x + 3 = 12$. Take care to keep all equations with positive coefficients and positive solutions. Keep inequalities positive.

Recall from Grade 6, Part I, p. 113 that they have seen "Find the rational number n which makes (the) sentence $\frac{2}{3} \times n = 1$ true". But they may not have seen, "Solve $3 \times n = 2$" or "$\frac{2}{3} \times n = \frac{4}{15}$", although they have divided rational numbers (Grade 6, Part II, p. 307ff). Note, Ibid, pp. 340-345 that in an "Exploration" section they consider $n \times \frac{2}{7} = \frac{5}{4}$ and on pp. 349-353 they have had a brief treatment for solving $\frac{2}{3} \times n = \frac{4}{5}$. *In particular the notation of juxtaposition for multiplication as in $3x = 12$ will have to be introduced*.

We should also consider the remarks of R.C. Buck about our use of equations and inequalities as tools for *inference*. This will be new to the student.

Show that $4x = 8$ if and only if $2x = 4$, or $4x = 8$ if and only if $x = 2$. Or, perhaps the chain of inference: $4x = 8$ implies that $4x - 4 = 8 - 4 = 4$ which implies that $4(x - 1) = 4$, which implies that $x - 1 = 1$, which implies that $x = 2$; and conversely. Various routes to the conclusion $x = 2$ should be encouraged and suggest a game for the students. How much logic should be included here is an open question.

Now, from the Grade 6 background (see also ISSM, Vol. I, Part 2, p. 195-196) the students are ready to see that, for example:

$\frac{4}{3}$ is the solution for $3x = 4$; $\frac{4}{3}$ means $4 \div 3$; $4 \div 3 = x$ means $3x = 4$

$\frac{1}{2}$ is the solution for $2x = 1$; $\frac{1}{2}$ means $1 \div 2$; $1 \div 2 = x$ means $2x = 1$

$\frac{3}{4}$ is the solution for $4x = 3$; $\frac{3}{4}$ means $3 \div 4$; $3 \div 4 = x$ means $4x = 3$

etc.

After some more examples try: If $b \neq 0$,

$\frac{a}{b}$ is the solution for $bx = a$; $\frac{a}{b}$ means $a \div b$; $a \div b = x$ means $bx = a$

as a pattern.

Now make the philosophical point that the rational $\frac{4}{3}$ is the number whose $\left\{\begin{matrix}\text{defining}\\ \text{key}\\ \text{primary}\end{matrix}\right\}$ property is that it is the solution of $3x = 4$. Repeat with special examples until exhausted and then with symbols a and b. (We should also do inequalities here. but we had better wait until a later section when we have discussed the result of multiplying or dividing an inequality by a positive number.)

Section 2. Arithmetic operations from the point of view of equations.

Here is a somewhat fresh approach to addition and multiplication of rational numbers utilizing their properties as solutions of equations. Since

$$\frac{4}{3} \text{ is the solution of } 3x = 4$$

and

$$\frac{1}{2} \text{ is the solution of } 2y = 1,$$

$\frac{4}{3} + \frac{1}{2}$ should be the solution of some equation of the form $bz = a$.
If $3x = 4$ then $2 \cdot 3x = 2 \cdot 4$ --- or $6x = 8$
If $2y = 1$ then $3 \cdot 2y = 3 \cdot 1$ --- or $6y = 3$.
By addition we conclude that $6x + 6y = 8 + 3$ or that $6(x + y) = 11$ -- which is an equation of the desired form, i.e., the solution of $6z = 11$ which is $\frac{11}{6}$ should be $x + y$, or $\frac{11}{6} = \frac{4}{3} + \frac{1}{2}$. Compare with the conventional algorithm: $\frac{4}{3} + \frac{1}{2} = 4 \cdot \frac{2}{3} \cdot 2 + 3 \cdot \frac{1}{3} \cdot 2 = \frac{8}{6} + \frac{3}{6} = \frac{11}{6}$.
A similar, but easier, trick provides the algorithm for multiplication: From $3x = 4$ and $2y = 1$ we get, by multiplication $(3x)(2y) = 4 \cdot 1$, or $6(xy) = 4$, or $\frac{4}{3} \cdot \frac{1}{2} = \frac{4}{6}$.

With these operations at our disposal proceed to solve some mathematical sentences.

Section 3. Order -- See ISSM Vol. I, Part 2, p. 276ff.

This is a good place to insert some story problems to help heighten the interest in comparing rational numbers. Here are a couple?

1. You are the manager of a baseball team. You need a new shortstop. You can trade for Willie Much or Mickey Little, both of whom appear to be equally good glove men. In previous play, Willie Much has come to bat 225 times and has 53 hits. Mickey Little has come to bat 183 times and has 43 hits. On the basis of this information, which would you choose? (An analysis of this information, which would you choose? (An analysis of this problem might well include whether there is any significant difference in these players anyway.)

2. You go to a picnic and are invited to join either of two tables. At Table A there are now sitting 7 people with 5 quarts of ice cream. At Table B there are sitting 10 people and 7 quarts of ice cream. At which table will your share of the ice cream be the greater? (Be careful, if you compare $\frac{5}{7}$ and $\frac{7}{10}$, you haven't counted yourself!) Introduce the notation $a < b$ and its equivalent $b > a$. (Point out that the "big side of the wedge is near the bigger number".) Verify the transitive property: $a < b$ and $b < c$ implies $a < c$. Use this to compare $\frac{5}{8}$ and $\frac{6}{14}$ via $\frac{5}{8} > \frac{4}{8} = \frac{1}{2} = \frac{7}{14} > \frac{6}{14}$. Compare fractions with equal denominators and then with equal numerators. Derive the decision method for $\frac{a}{b} < \frac{c}{d}$ $(ad < bc)$. Point out that from $ad < bc$ we can infer $\frac{a}{b} < \frac{c}{d}$.

   Discuss density: $a < b$ implies $a < \frac{(a + b)}{2} < b$. Consider problems like:

   Find three numbers between $\frac{2}{3}$ and $\frac{3}{4}$.

   Discuss the function: $x \to \frac{1}{x}$. Show that $a < b$ implies $\frac{1}{a} > \frac{1}{b}$.

   Show that the ray $[1,\infty]$ gets squeezed into $(0,1]$.

Prove: $a < b$ if and only if $a + c < b + c$.

$a < b$ implies $ac \leq bc$ and if $c > 0$ then $ac < bc$.

Mathematical sentences which are inequalities.

Graphing inequalities in one and two dimensions.

Do ones of the form $a < x < b$ and $c < y < d$,

and $\{(x,y) : x + y < 1 \text{ and } x \geq 0 \text{ and } y \geq 0\}$,

and $\{(x,y) : y > x \geq 0\}$.

Section 4. Decimals -- Compare with Grade 6, Part I, p. 59ff.

Motivate need for decimals by the desire to have an easy method for comparing rationals. Return to problem of finding several numbers between $\frac{2}{3}$ and $\frac{3}{4}$.

Expanded notation, using positive exponents (ISSM, Vol I, Part 2, 309-314)

Extension to negative exponents (ISSM, Vol II, Part 2, 386-388)

Scientific Notation (ISSM, Vol II, Part 2, Chapter 21, 379-394)

Rationals as Repeating Decimals (ISSM, Vol I, Part 2, 325-334)

Section 5. Percent.

Practice computations of the form "Find a% of b". Point out that "of" means multiplication here.

GRADE 7 - CHAPTER 3

(Alternate Version)

THE SET OF RATIONALS

Background Assumptions:

1. Students have the complete set of integers at their disposal and have had extensive work with the non-negative rationals.

2. Operations introduced thus far are:

   Addition, subtraction, multiplication, division for non-negative rationals.
   Addition and subtraction for negative integers.

3. Other ideas already introduced included:

   Use of variables as names of numbers.
   Informal solution of sentences (missing addend, etc.).
   Ordering of integers and positive rationals.
   Use of multiplication property of one for changing form of a fraction.
   Decimal estimates for numbers named by fractions.
   Simplifying complex fractions.

Rationale:

1. The negative rationals will be introduced by the opposite function in order to complete the set of rationals as soon as possible. The negative rationals have been left "dangling" since the student was introduced to the negative integers in Grades 5 and 6.

2. If the complete set of arithmetic operations (addition, subtraction, division, and multiplication) is introduced early for the set of rationals, these skills can be used and reinforced throughout the rest of Grades 7, 8, and 9.

3. The rational number system will be "endowed" (informally) with the familiar properties that the student has been working with in the set of positive rationals. Special note will be made of the fact that every rational number has a unique additive inverse, and a unique multiplicative inverse. It is recommended that uniqueness not be proved here, only a statement that it can be proved, and then plan to prove it in later grades when the discussion of structure becomes more formal.

Purposes:

1. To provide a meaningful review of operations with integers and positive rationals, and extend the operations of +, x, -, ÷, to the negative rationals and to simple expressions containing variables.
2. To review, early in the 7th year, the basic properties of the integers and the positive rationals and extend, informally, these properties to the set of rationals.

Section 1. The Opposite Function.

1.1 Opp: $x \rightarrow -x$. Tie in with student's knowledge of the integers. (Refer to Chapter 2, Graphing, Function, and Variables). There should be little difficulty convincing the student of the existence of the opposite of $\frac{2}{3}$ such that $\frac{2}{3} \times (-\frac{2}{3}) \div 0$. (Use reflections about the origin on the number line as in Grade 9, First Course in Algebra, Part 1.)

Note that we should simply point out again that the "-" is now being used to indicate a new unary operation. The connection between "oppositing" and subtracting should be made clear when appropriate.

1.2 Establish $-(-x) = x$. (See p. 109, FCA, Part 1)
This will be handy to have when we discuss subtraction.
Some appropriate exercises are in the above reference.

Section 2. Properties of Rational Numbers.

2.1 The following properties are familiar and have been stated for the positive rationals, and the integers. We will assume that there exists operations of addition and multiplication such that the old operations defined previously for some of the rationals still hold (closure, commutativity, associativity, multiplicative identity, additive identity, additive inverse, multiplicative inverse, distributivity.) This section can provide a light review or arithmetic operations with the positive rationals, combined with a light reinforcement of the structure developed so far.

2.2 In this section some attention should be given specifically to the fact that every rational number has an additive inverse. (See FCA, Chapter 5)

Note: It is recommended that the raised dash, $^-3$, not be introduced. This is awkward notation with rational numbers in fraction form, and the symbol is dropped soon after it is introduced. It is suggested that the language of the students be carefully developed so that he reads (-3) as the "negative of 3", "the opposite of 3", or "the additive inverse of 3". The student should be discouraged in the beginning from reading (-3) as "minus 3" or "negative 3". "Minus" tends to confuse the issue with the operation of subtraction, while "negative" tends to become confusing when variables are introduced.

Section 3. Absolute Value Function.

3.1 Definition and practice with definition. $f : x \rightarrow \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$ (See FCA, pp. 113-116)

3.2 Distance between a rational number and 0 on the number line is the absolute value of that number. (See FCA, pp. 113-117) This will be helpful in establishing the "sign" of the number which is the result of an addition.

Section 4. Addition and Multiplication of Rational Numbers.

4.1 Review addition of Integers. (See FCA, Part 1, pp. 121-129. Also Grade 6, Chapter 4, pp. 193-236, and Math. For JHS, Vol. 2, Part 1, pp. 12-21)

4.2 Addition of Rational Numbers as a natural extension of the addition of integers as done on the number line in Grade 6.
Review addition of positive rationals, then do addition of all rationals. Point out that with absolute value we can now duplicate "algebraically" the "arrow" routine on the number line which becomes tedious for rational number addition.
Discuss the additive inverse of a rational number, graph on the number line, and informally convince the student of the existence and uniqueness.

4.3 Multiplication of Rational Numbers

Develop first intuitively as in Math. For JHS, Vol. 2, Part 1, pp. 34-38. Also consider using rate of temperature increase or decrease, or increase and decrease of the area of a rectangle. Also use graph $y = -2x$ to motivate multiplication of negative rationals. Use absolute value notation. (See FCA, Part 1, pp. 146-147)
Prove: $-a = (-1)a$ (pp. 155-156, FCA, Part 1) for all rational numbers a.
Discuss and graph on the number line the multiplicative inverse of the rational numbers. (See pp. 162-164, FCA, Part 1) The multiplicative inverses of -1, and 1 should have particular attention.
Note: Throughout all of these discussions rational numbers in fraction form should be represented "$-\frac{a}{b}$" rather than $\frac{-a}{b}$, or $\frac{a}{-b}$. We can handle these situations later.

Section 5. Division of Rational Numbers.

5.1 Definition of division. "a divided by b", means a times the multiplicative inverse of b, $b \neq 0$. (See FCA, Part 1, pp. 223-227)

This definition should cause no trouble with $\frac{-5}{15}$ or $\frac{5}{-15}$ since we will have established in the previous section that the multiplicative inverse of -15 is $-\frac{1}{15}$.

5.2 Prove: $\frac{a}{a} = 1$ for all rational numbers $a \neq 0$. In particular we need $\frac{-1}{-1} = 1$ to simplify complex fractions. (See MJHS, Vol. 2, Part 1, p. 45 for exercises, and FCA, Part 1, pp. 224-229, also pp. 234-241)

Properties of the fraction form of the rational number. (FCA, Part 1, pp. 233-235)

Section 6. Subtraction of Rational Numbers.

6.1 Review subtraction of integers from Grade 6.

Definition: $a - b$ means $a + (-b)$. Be sure to make clear that here we have a double use of the "-" sign. (See FCA, Part 1, p. 210)

Remind students of the statement $-(-x) = x$ developed in Section 1.

Establish $-(a + b) = (-a) + (-b)$.

6.2 Practice in subtraction as in FCA, Part 1, pp. 211-218.

6.3 Subtraction in terms of distance on the number line as in FCA, Part 1, pp. 219-222.

GRADE 7 - CHAPTER $3\frac{1}{2}$

(Alternate Version (continued))

THE SET OF RATIONALS; SOLUTION OF MATHEMATICAL SENTENCES

Background Assumptions:

1. Multiplication, addition, division, subtraction or rational numbers.
2. Ordering of integers and of positive rationals.

Rationale:

1. This chapter will allow the student to use his newly acquired skills in operating with the rationals while he learns a little about the process of finding solution sets of mathematical sentences.
2. Properties of order and equality will be discussed and reviewed.

Purposes:

1. To prepare for a formal approach to the solution of equations and inequalities.
2. To review decimal notation for rational numbers and to discuss the density of the rationals.

Section 1. Decimal names for Rational Numbers.

1.1 Expanded notation using positive exponents. (ISSM, Vol. 1, Part 2, pp. 309-314)

1.2 Extension to negative exponents. (ISSM, Vol. 2, Part 2, pp. 386-388)

1.3 Scientific Notation (ISSM, Vol. 2, Part 2, Ch. 21, pp. 379-394)

1.4 Rationals as repeating decimals. (ISSM, Vol. 1, Part 2, pp. 325-334)

Section 2. Ordering the Rationals.

2.1 Basic properties of order. (9H, pp. 186-196)
From notation: $a > b$ is the same as $b < a$.
"Comparison" property (9H, p. 191)
Transitive property (9H, p. 193)

2.2 Addition and Multiplication properties of Order.
See FCA, Part 1, pp. 187-190, and pp. 195-197.
Prove: if $x \neq 0$, then $x^2 > 0$.

2.3 Order of rationals.
Discuss order of rationals on the number line.

2.4 Density of rationals.

Section 3. Introducing percent.

3.1 "%" as a function. Define $\%: x \rightarrow \frac{x}{100}$. Hence the symbol % will mean "multiply by $\frac{1}{100}$" or "multiply by .01".

3.2 Practice computation. "Find a% of b", etc.

Section 4. Solutions of Equations and Inequalities.

4.1 Restatement of properties of equality and order as in FCA, Part 1, pp. 204-205.

4.2 Solutions of equations and inequalities of the form

$$x + a = b, \quad ax = b, \quad ax + b = c, \quad \text{and} \quad ax + bx = c,$$

and inequalities of similar form.

Informal use of the properties of rational numbers, properties of equality, and properties of order. The notion of equivalent sentences should be started here, but not developed in full.

Graphs of solution sets should be done also.
Applications to verbal problems and simple translations required.
Ample practice with rationals in fractional and decimal form should be provided.

GRADE 7 - CHAPTER 4

CONGRUENCE (REPLICATION OF FIGURES)

Background:

Congruence of segments, angles, triangles, copying, etc.: Grade 5, Part I, Chapter 4. Measure of segments and angles: Grade 4, Part II, Chapter 9; Grade 5, Part II, Chapter 7.

Purpose:

1. Review above material.
2. Introduce SAS and ASA motivated by the corresponding copying problem (SSS to be reviewed).
3. Give practice in making elementary deductions based on the triangle properties.
4. Begin to refine the vague idea that "congruent figures can be made to coincide" by introducing specific types of motions (slide, turn, flip),
   (a) physically,
   (b) intuitively,
   (c) in a grid or coordinate framework.

Rationale:

We continue the process of structuring or modeling space, begun in Chapter 1. We conceive space not merely as structured in terms of the points, lines and planes that fill up every part of it but as being "homogeneous", as being everywhere the same. This vague language indicates a key property of Euclidean geometry (and incidentally of the classical non-Euclidean geometries): Figures can be copied freely in space. The concept of replication is studied formally in this chapter for triangles, but its wider application is indicated at least intuitively by exercises in which other types of figures appear.

A related concept: Ruler and compass procedures enable us to replicate the most fundamental figures, particularly triangles, but they yield little for replication of figures in general. For this purpose we need the thread of motions (or isometries). A motion operates on any figure to produce a congruent figure. We start to weave this idea into our structure by introducing three types of isometry: translation, rotation, reflection (in a line?).

Outline

1. Congruence of segments, of angles.

Review and refine, emphasizing that two segments (or two angles) are congruent if one is a copy of the other. Briefly review the concrete process for copying segments and angles by ruler and compass.

Connections Between Congruence and Incidence Properties

Exercise 1: Given $\overline{AB} \cong \overline{BC}$, can A, B, C be on a line? Must A, B, C be on a line? Can A, B, C be in a plane? Must they be in a plane? Why?

Exercise 2: Similarly for $\overline{AB} \cong \overline{BC}$, $\overline{BC} \cong \overline{CD}$.

Exercise 3: Similarly for $\overline{AB} \cong \overline{BC}$, $\overline{BC} \cong \overline{CA}$.

Exercise 4: Similarly for $\overline{AB} \cong \overline{BC}$, $\overline{BC} \cong \overline{CD}$, $\overline{CD} \cong \overline{DA}$.

Exercise 5: Similarly for $\overline{AB} \cong \overline{BC}$, $\overline{CD} \cong \overline{DA}$.

Exercise 6: Suppose $\overline{AB} \cong \overline{CD}$ and A, B, C, D are on one line. Must $\overline{AC} \cong \overline{BD}$? Can it? When?

Exercise 7: Suppose $\overline{AB} \cong \overline{CD}$ and $\overline{AC} \cong \overline{BD}$. Can A, B, C, D be on a line? Must they be?

Exercise 8: Suppose O, A fixed and $\overline{OA} \cong \overline{OX}$. Where can X lie?

Try similar problems for congruence of angles, e.g., involving $\angle AOB \cong \angle BOC \cong \angle COA$, etc.

2. Division of segments and angles into two congruent parts. Define midpoint of a line segment and angle bisector.

Exercises: A set of finger exercises similar to those given in Section 1.

A set of paper folding exercises along the lines of the following:

Give 5 triangles to be used as patterns (make them rather large). They might be an acute scalene triangle, an obtuse scalene triangle, a right scalene triangle, an obtuse isosceles triangle, and an equilateral triangle.

Note: They are not familiar with this terminology. Just give them as a "random" set of triangles. Label each triangle by a single capital letter in the interior of the triangle (A,B,C,D,E).

Directions:

(1) Make two copies of each of these (by tracing, or by ruler and protractor or by straight edge and compasses). Label the interior of each copy by the letter corresponding to the given figures and cut the triangles out.

(2) A median of a triangle is a line segments that joins a vertex of a triangle to the midpoint of the opposite side. How many medians does a triangle have? Take one of your triangles that is labeled A and follow these steps:

   (a) Fold to find the midpoint on one side. Do not fold the whole paper. Just pinch the paper to indicate the midpoint.

   (b) Now fold the triangular region to show a median of the triangle.

(c) Repeat (a) and (b) for each of the other two sides of the triangle.

Now follow these three steps for each of the triangles labeled B, C, D, and E.

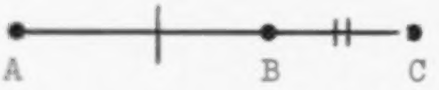
(3) An angle bisector of a triangle ...

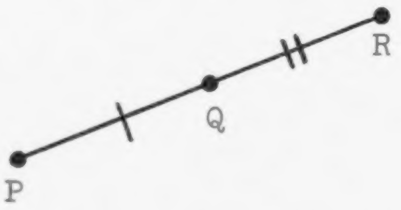
(continue in manner of (2))

Note: Given the background of the students, it does not seem advisable to fold the altitudes and perpendicular bisectors of the sides at this time.

3. Addition Property for Segments.

Suppose $\overline{AB} \cong \overline{PQ}$, $\overline{BC} \cong \overline{QR}$, B is between A and C, and Q is between P and R. Then $\overline{AC} \cong \overline{PR}$. (Note: betweenness was introduced in Chapter 1.)





Motivated (a) by using sticks; (b) by applying compasses to test $\overline{AC} \cong \overline{PR}$ in a drawing.

Exercise: Suppose $\overline{AB} \cong \overline{PQ}$, $\overline{BC} \cong \overline{QR}$. How are $\overline{AC}$ and $\overline{PR}$ related?

(Accept answers using sticks (pencils), drawings.)

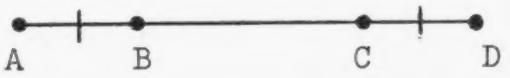
Another version: Suppose $\overline{AB} \cong \overline{PQ}$, $\overline{BC} \cong \overline{QR}$. Must $\overline{AC} \cong \overline{PR}$? Can $\overline{AC}$ be greater than $\overline{PR}$? less than $\overline{PR}$?

Query: Maybe last problem should be motivation for the topic.

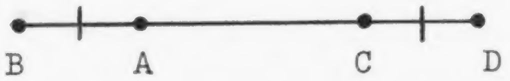
Warning: Additivity is a simple property and intuitively very familiar; it is merely one part of the geometrical postulate "If equals are added to equals, the sums are equal." Don't let its essential simplicity get lost in the verbiage. Maybe refer to the underlying idea as "addition of sticks".

Exercise 1: In figure, given $\overline{AB} \cong \overline{CD}$, what can you conclude?

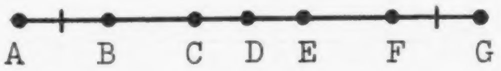
(a)



(b)

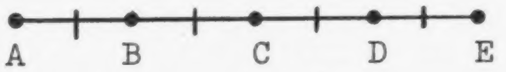


Exercise 2: In figure, given $\overline{AB} \cong \overline{FG}$, what can you conclude?

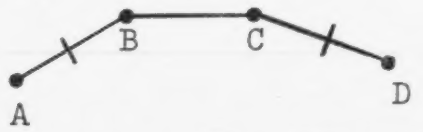


(Note: make $\overline{CD} \cong \overline{DE}$ and $\overline{BC} \cong \overline{EF}$ in the diagram but do not make this given information to see if the students are inferring more than betweenness from the diagram.)

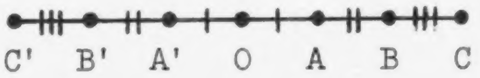
Exercise 3: Similarly given $\overline{AB} \cong \overline{BC}$, $\overline{BC} \cong \overline{CD}$, $\overline{CD} \cong \overline{DE}$.



Exercise 4: Similarly given $\overline{AB} \cong \overline{CD}$.



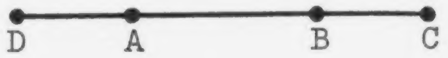
Exercise 5: Similarly given $\overline{OA} \cong \overline{OA'}$, $\overline{AB} \cong \overline{A'B'}$, $\overline{BC} \cong \overline{B'C'}$.



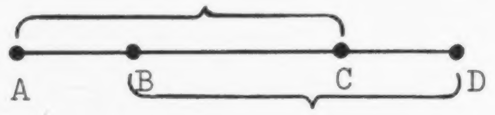
4. Subtraction Property for Segments.

Suppose $\overline{AC} \cong \overline{PR}$, $\overline{AB} \cong \overline{PQ}$, B is between A and C, and Q is between P and R. Then $\overline{BC} \cong \overline{QR}$. Treat lightly!

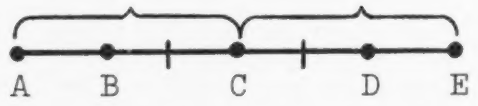
Exercise 1: In figure, given $\overline{AC} \cong \overline{BD}$ what can you conclude?



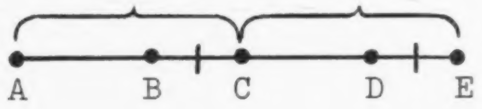
Exercise 2: What can you conclude from (information indicated in) figure?



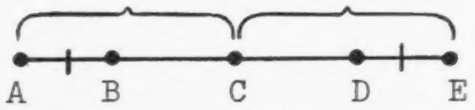
Exercise 3: What can you conclude:



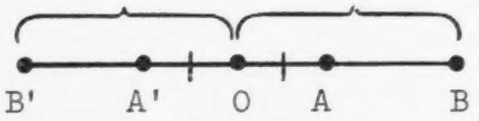
Exercise 4: What can you conclude?



Exercise 5: The same for



Exercise 6: The same for



(Note: This might suggest reflection of the line in point O. Will the students see it as the same exercise as Exercise 3?)

5. Addition and Subtraction Property for Angles.

Make this a parallel development to Sections 3 and 4; only do it in one section.

Statement of Addition Property for Angles.

Suppose $\angle ABC \cong \angle PQR$, $\angle CBD \cong \angle RQS$, $\overrightarrow{BC}$ is between $\overrightarrow{BA}$ and $\overrightarrow{BD}$, and $\overrightarrow{QR}$ is between $\overrightarrow{QP}$ and $\overrightarrow{QS}$. Then $\angle ABD \cong \angle PQS$.

Treatment similar to Sections 3 and 4.

Note: Betweenness of rays is to be introduced in Chapter 1.

6. Vertical Angles.

Try to use definition in terms of opposite ray. Also stress other form (see MJHS, Vol. 1, p. 398).

Query: Introduce opposite ray in Chapter 1 or here?

Geometric Property: Two vertical angles are congruent.

Motivation: Construct a pair of vertical angles on cardboard, cut out the angular regions and try to make them fit. Use scissors or sticks to form dynamic models of vertical angles.

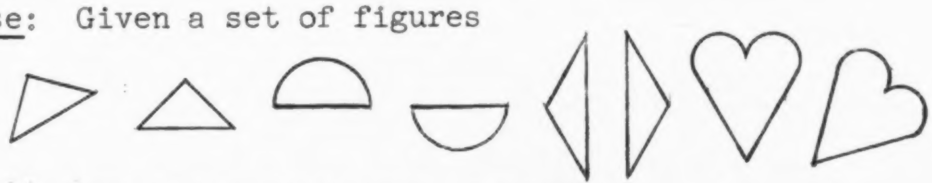
Give practice in picking out pairs of vertical angles. *

Example: Find angle vertical to a given one. How many pairs of vertical angles are formed?

7. The Concept of Congruence.

Review the concept for figures in general. Find examples in classroom; don't restrict to planar figures. Please make examples of wire, cardboard, etc. Include some reference to modeling: what is an exact copy; under mass production do we get exact copies. Might refer to such things as printing presses, boxes of screws, printed T.V. circuits, automobiles, etc. See also the chapter on Empirical Validity in Geometry in Basic Concepts of Geometry by Prenowitz and Jordan (publisher, Blaisdell).

Exercise: Given a set of figures



(a) Find pairs which are congruent.

(b) Discuss how to make them coincide by slide, turn, flip.

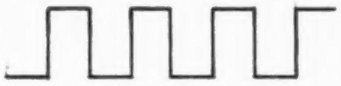
Include in this exercise examples that are not triangles, some wire models, some non-planar cardboard figures, etc.

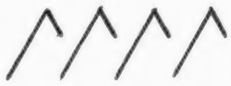
Note: Possibly use color and keep examples simple.

8. Congruence of a Figure with Itself.

Stress identity congruence as well as non-identity.

Work with isosceles triangles, equilateral triangles, rectangles, circles, squares, hearts (cardioids to you!), scalene triangles, regular polygons. Include some unbounded figures, friezes, etc., wall paper patterns; tilings of a plane.





Refer to axes of symmetry; illustrate and discuss in class: reflection symmetry, rotation symmetry, translation symmetry.

Exercises in finding axes of symmetry.

Exercise: Find the symmetries of each of the letters of the alphabet. Perhaps standardize by using the capital letters on a typewriter. Students could even group the letters according to their symmetries. See manuscript "Mathematics of the Alphabet" submitted by Ranucci to SMSG Panel on Supplementary Publications.

Exercise: List the motions (symmetries) of a simple figure, e.g., equilateral triangle, rectangle, square. What happens if you combine two of these motions, that is, follow one by another?

9. Congruence of Triangles.

Review and refine definition; emphasize correspondence between sets of vertices. Thus $\triangle ABC \cong \triangle PQR$ means $\overline{AB}$, $\overline{BC}$, $\overline{AC}$, $\angle ABC$, ... $\cong$ $\overline{PQ}$, $\overline{QR}$, $\overline{PR}$, $\angle PQR$, ...

Practice in finding corresponding parts when certain ones are given.

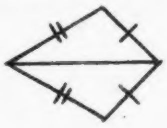
10. The SSS Congruence Property of Triangles.

Use a written set of class exercises based on the construction, of triangles given the three sides (See Grade 5, Chapter 4, pp. 194-195). Have students compare results and see the congruence of the triangles obtained.

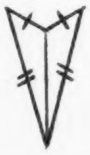
State and discuss the SSS congruence property.

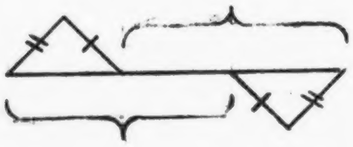
Develop the construction for the bisector of an angle.

Exercise: What can be deduced from the given figure?



Exercise: Similarly these:





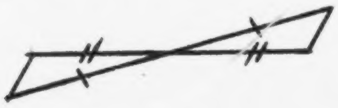
Exercise: Construct the three angle bisectors of several given triangles.

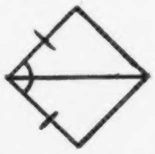
11. The SAS Congruence Property of Triangles.

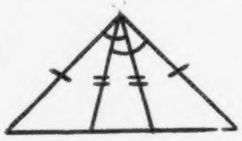
Use a written set of class exercises based on the construction of triangles given two sides and the included angle. Have students compare the results and see the congruence of the triangles obtained.

State and discuss the SAS Congruence property.

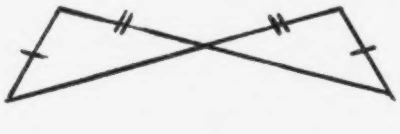
Exercise: What can be deduced from each of the following figures?

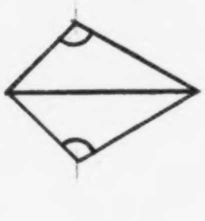






Include some where the order of SAS is not found:





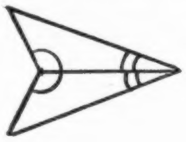
Deduced Property: If two sides of a triangle are congruent then the opposite angles are congruent.

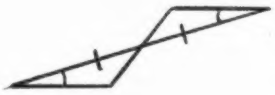
12. The ASA Congruence Property of Triangles.

Use a written set of class exercises based on the construction of triangles given two angles and the included side. Have students compare their results and see the congruence of the triangles obtained.

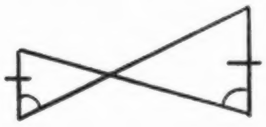
State and discuss the ASA Congruence Property.

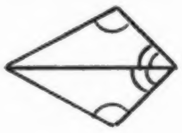
Exercise: What can be deduced from the given figure?





Include some where the side is not included:





Deduced Property: If two angles of a triangle are congruent then the opposite sides are congruent.

13. Motions by means of a Co-ordinate system.

13.1 Sliding (translation)

Discuss sliding as a physical process. Slide a wire triangle on a table top. Try giving different directions that still cause a slide. Build up to the "neatness" of co-ordinatizing the situation.

Exercises: A series of problems with a given finite point set and a mapping that determines a translation.

Example:

$A = (1,1),\quad B = (1,2),\quad C = (3,2)$

Map $A \to A',\quad B \to B',\quad C \to C'$ if

(a) $(x,y) \to (x + 3 , y)$

(b) $(x,y) \to (x , y + 4)$

(c) $(x,y) \rightarrow (x + 3, y + 4)$

For each of these, plot A, B, C and A', B', C'.
Draw △ABC and △A'B'C'. How are these triangles related?

Have the students orient one triangle and see if the orientation (see Chapter 1) has been changed.

13.2 Turning (rotation)

Discuss turning as a physical process. Turn a wire triangle on a table top. Try giving different directions that still cause a turn. Emphasize a fixed point for each turn.

Exercises: A series of problems where the students are given two finite point sets that are related by a rotation. Have them plot the point sets and draw the figures. Let them try to find how the second point set was derived from the first.

Example:

$$\begin{tikzcd}
 & \text{(a) } A' = (-1,-1),\ B' = (-1,-2),\ C' = (-3,-2) \\
A = (1,1),\ B = (1,2),\ C = (3,2) \arrow[ur] \arrow[r] \arrow[dr] & \text{(b) } A' = (-1,1),\ B' = (-2,1),\ C' = (-2,3) \\
 & \text{(c) } A' = (1,-1),\ B' = (2,-1),\ C' = (2,-3)
\end{tikzcd}$$

For each of these, plot A, B, C and A', B', C'.
Draw △ABC and △A'B'C'. How are these triangles related?

Have the students check to see if the sense of orientation has changed. Have students locate fixed points of each turn.

13.3 Flipping (reflection).

Discuss flipping as a physical process. Flip a wire triangle on a table top. Try choosing many different lines for the axis of the flips. Discuss the possibility of doing this by coordinatizing the plane in a "neat "manner (make the axis a coordinate axis).

Exercises: Again use a series of problems where only the vertices are actually transformed and then the figures are drawn:

(a) $(x,y) \rightarrow (x,-y)$;
(b) $(x,y) \rightarrow (-x,y)$;
(c) $(x,y) \rightarrow (y,x)$;
(d) $(x,y) \rightarrow (-y,-x)$.

Note the change in orientation under this transformation. Perhaps give a brainbuster which involves a slide reflection. Possibly include a problem where axis of reflection is $x = 2$ or $y = -6$.

GRADE 7 - CHAPTER 5

MEASURE

Purpose:

To introduce a metric for geometry. To review and extend the concepts of linear measure and angular measure and extend to arc measure. To work with the equivalence of polygonal regions within a plane on a non-numerical basis.

Rationale:

1. To begin further work with measure concepts early in the 7th grade.
2. To provide the tools to move from congruence to similarity.
3. To begin to develop the interrelations of measure and congruence although this requires the real numbers for its full development.
4. To begin the Pythagorean Property for use in several other chapters although it is hoped that this property will be re-investigated as the students develop irrational numbers and the algebraic skills for some alternate proofs of the property.

Background:

Extensive work with linear and angular measure is presently done in Grades 4, 5 and 6. The terminology of nonmetric geometry. Congruence of segments and polygonal regions. They have NOT had irrational numbers of any form ($\sqrt{2}$, $\sqrt{7}$, π, etc.)

Section 1. Linear Units of Measurement.

1.1 Linear Units of measurement

A careful redevelopment of what is meant by linear units of measurement. The last time this was done was in Grade 4, Chapter 9. (It was reviewed briefly in Grade 5, Chapter 4, pp. 407-409.) The concept needs to be redeveloped along the lines of MJHS Vol. 1, pp. 249-251, 260-274 or ISSM Vol. 2, pp. 1-32. However, much less detail and expansiveness should be necessary due to their background. Be sure to use the congruence concepts developed in Chapter 4. Extend the concept of congruent segments to the idea that congruent segments have equal measure. Discuss the idea that every line segment has a measure. Tread (but lightly) on the idea that this measure often is a number familiar to them, but that the measures of many line segments are numbers which they have not studied yet. Until they meet these numbers, they can only give such measures by approximation.

1.2 Applications of Linear Units.

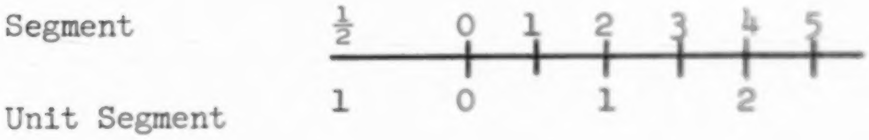
Use of concept developed in 1.1. See the same reference for many ideas. Add such things as the amount of wire needed to make a 3-inch cube, etc. Might end with a "discovery" exercise for the "perimeter" of a circle. Use some additivity of segments by measure (see last chapter for additivity using congruence). Perhaps some WST exercises on the Archimidean Property (see below).

Consider the number line (with or without negative numbers):

Somewhere on this line appears a point with the coordinate $\frac{172}{39}$. Even though we cannot place it on the number line as we have started it above, let us think about it on the line and answer some questions:

(a) Continue marking off congruent unit segments from zero and label the points they determine with the numbers 1, 2, 3, 4, 5, etc. What is the number on the point just before $\frac{172}{39}$? Just after it? Which is closer?

(b) Start again with a segment of half the length of the unit. Step it off on the number line and mark the successive steps 1, 2, 3, etc., thus:



What is the label on the mark just before $\frac{172}{39}$? Just after it? Which is closer?

(c) Repeat the process of (b) with a segment which is 0.1 the size of the unit. Again, with a segment 0.01 the length; and then 0.001 the length.

1.3 Linear measure and circles.

A brief review of circles and segments associated with circles (see Grade 4, Chapter 5 and Grade 6, Chapter 9). An initial development of the circumference of a circle and the number π (see MJHS Vol. 1, pp. 491-500 and ISSM Vol. 2, pp. 181-187).

Exercise: (Please get some real facts!)

The park ranger stated that this stand of redwoods has an average diameter of 4.2 feet when measured 6 feet from the ground. The Colonel Molotov Tree has a diameter of 8.3 feet when measured in the same way. Lots of "modeling" involved here. Might have them find such things as: the average circumference of the trees; the lengths of fences (to thwart souvenier hunters) if the fences are 5 feet from the trees; the lengths of semi-circular walks around the trees, etc.

Section 2. Angular and Arc Measure.

2.1 Angle measure.

This has been carefully developed in Grade 5, Chapter 7 and reviewed slightly in Grade 6, Chapter 3. The development at this time is essentially review but important. The vocabulary for angles should be introduced (acute, right, obtuse). For reference, see MJHS Vol. 1, pp. 287-295 or ISSM Vol. 2, pp. 75-79, 90-94.

Use some additivity of angles by measure.

2.2 The sum of the measures of the angles of a triangle.

This has been introduced as a property in Grade 6, Chapter 3. It is needed for further work this year and should be restated and its implications more carefully investigated.

Exercises:

(1) An obtuse triangle is a triangle with one obtuse angle. What can you say about the masures of the other angles?

(2) If the vertex angle of an isosceles triangle is 83°, what is the measure of each of the other angles?

(3) If one angle of an isosceles triangle is 70°, what is the measure of each of the other angles?

(4) If one angle of an isosceles triangle is 104°, what is the measure of each of the other angles?

(5) An acute triangle is a triangle that contains three acute angles. Could an acute triangle have two angles of 20° and 35°?

(6) If one angle of an acute triangle is 15°, what can you say about the measure of each of the other angles?

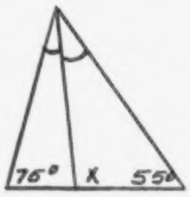
(7) A right triangle is a triangle that contains one right angle. Could a right triangle contain two right angles? Could a right triangle contain an obtuse angle? What can you say about the other two angles of a right triangle?

(8) If one angle of a right triangle is 23°, what is the measure of each of the other angles?

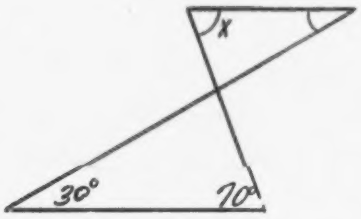
(9) What is the measure of each of the angles of an isosceles right triangle? An equilateral triangle?

(10) Find x: (Note: they do not have linear pairs yet)

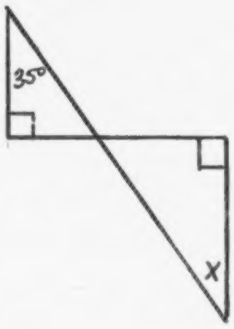
(a)



(b)



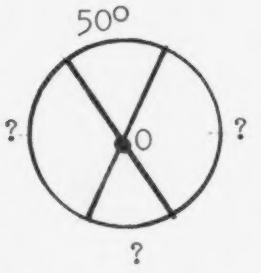
(c)



2.3 Arc measure and central angles.

The terminology of arc and central angle has been started in Grade 6, Chapter 9, pp. 567-576 and is reviewed in MJHS Vol. 1, pp. 481-486. The concept of an arc degree is new and might be developed in the manner of MJHS Vol. 1, pp. 487-495. Be sure to introduce semi-circles, quarter circles, etc.

Exercise:



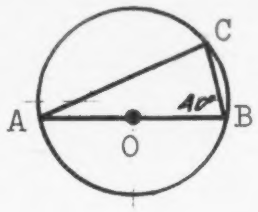
Teacher suggestion for Sections 2.4 and 2.5:

Make a circle nailboard by placing nails around a circle every 10 or 15 degrees. Use rubber bands or string to help the students see inscribed angles and the invariance of measure under certain conditions.

2.4 A triangle inscribed in a semi-circle.

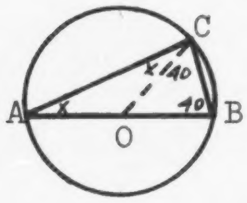
This section might be a set of exploratory exercises of this nature:

Given: ABC inscribed in circle O with O lying on $\overline{AB}$, $\angle B = 40^\circ$



Problem: Find the measure of $\angle C$.

Solution:



$$x + x + 40 + 40 = 180$$
$$2x = 100$$
$$x = 50$$
$$\angle C = x + 40 = 90$$

Repeat this basic problem for many different $\angle B$'s. Try to get to a student statement of the property: A triangle inscribed in a semi-circle is a right triangle.

2.5 Angles inscribed in circles.

Begin with a set of class exercises to "discover" the measure of an inscribed angle in terms of its intercepted arc.

Example Exercise:

(1) Draw a circle and with a protractor draw a central angle AOB such that $\widehat{AB} = 40^\circ$.

(2) Choose any point C on the circle and draw ∠ACB. Measure ∠ACB with your protractor.

(3) Choose any other point D on the circle and draw ∠ADB. Measure ∠ADB with your protractor.

(4) Repeat this procedure for at least four other points on the circle.

Be certain to crystalize what is meant by "an inscribed angle."

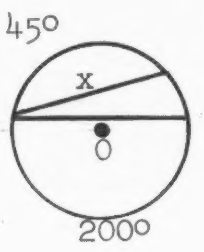
Property: The measure of an inscribed angle is one-half the measure of its intercepted arc.

Discuss that this is not true if the vertex of the angle does not lie on the circle.

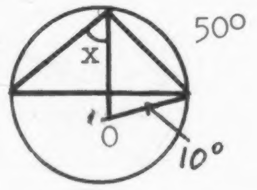
Exercises:

(1) Find **x**:

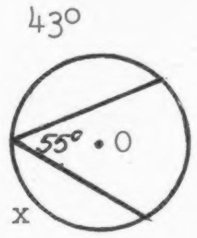
(a)



(b)

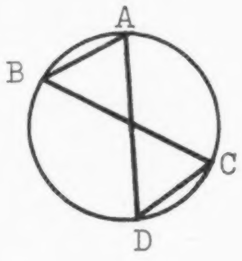


(c)



etc.

(2)

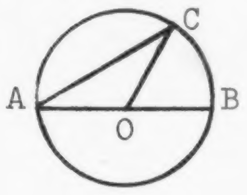


Is $\angle B \cong \angle D$? Try to explain why or why not.

(3) $\angle ABC$ is an inscribed angle and $\overset{\frown}{ABC}$ is a semi-circle. What is the measure of $\angle ABC$?

(4) Draw a circle O and a diameter $\overline{RS}$ of this circle. If A is any point on the circle, what is the measure of $\angle RAS$?

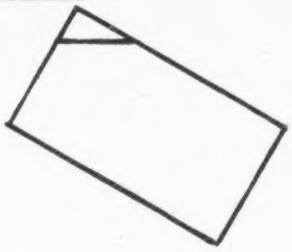
(5) Given: circle O with
diameter $\overline{AB}$
$\overset{\frown}{AC} = 110^\circ$



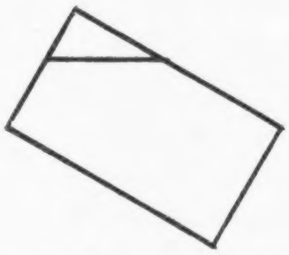
Find the measure of $\angle ACO$ and $\angle BOC$ and $\angle ACB$.

(6) Draw a 2 inch line segment. Take a piece of cardboard with a corner (right angle) and use it to draw as many right triangles as you can with the 2 inch segment as the hypotenuse. (Note: the hypotenuse of a right triangle is the side opposite the right angle.)

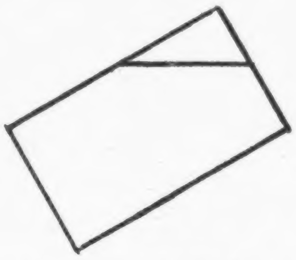
Example:



or



or



What do you observe? (probable answer: it gets crowded!)
(hopeful answer: the third vertex lies on a circle.)

Suggestion:

The problem might profitably be done in two stages:

(1) Draw the right triangles. What do you observe? (it gets crowded!)

(2) Repeat as in (1) but do not draw the right triangles. Just mark the position of the third vertex. What do you observe? (The third vertex lies on a circle.)

The Pythagorean Property and Applications.

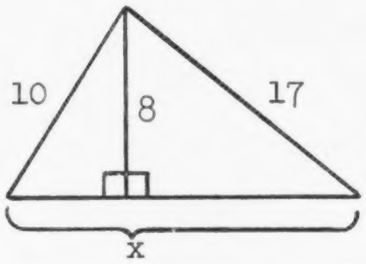
The background limitations here are formidable. However, this topic has been developed in MJHS Vol. 2, pp. 192-205 or ISSM Vol. 2, pp. 335-343.

In each of these references, the background limitations are essentially the same as for this chapter so the developments might well be comparable if these have been successful. Some additional applications are listed below (the references contain many good ones, too).

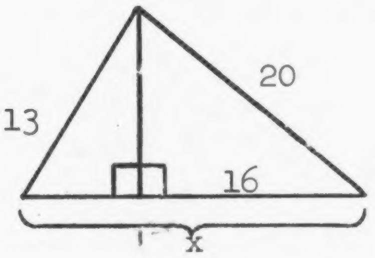
Exercises:

(1) Find x:

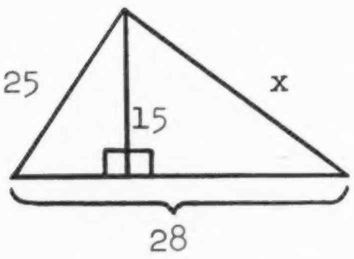
(a)



(b)

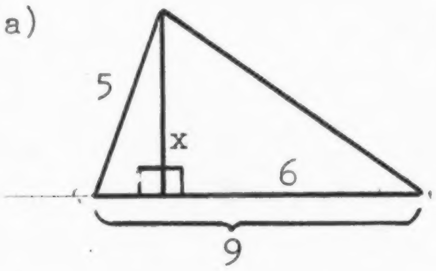


(c)

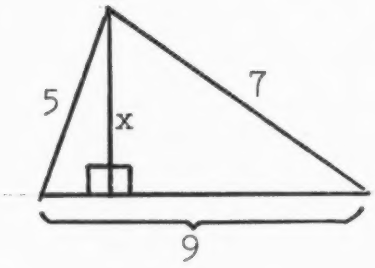


(2) Find x:

(a)



(b)



Note: The problem 2B probably cannot be solved by the students. Yet there is a unique solution. Might some find this interesting?

(3) A rectangle is a quadrilateral with four right angles. If ABCD is a rectangle and AB = CD = 7 and AD = BC = 24, find AC and BD.

(4) If ABCD is a rectangle and AC = 25 and BC = 15, find the perimeter of ABCD.

Section 4. Equivalence of Polygonal Regions.

Some definition for equivalence must be given. Keep it very simple. Possibly mention the relation to area measurement, but stress the idea that one region could be used to exactly "cover" the other.

4.1 Equivalent region building.

Intuitively develop feeling of a polygonal region in a plane, and discuss what is meant by two polygonal regions being equivalent but not necessarily congruent.

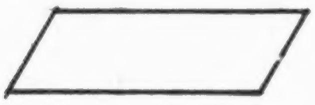
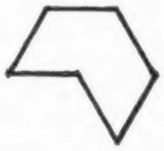
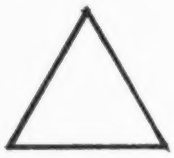
Develop the following problem carefully in the written text so that the student learns the rules of this particular set up.

Take four congruent equilateral triangular regions. Lay them on your paper to form a polygonal region following these rules:

(1) No triangular regions overlap.

(2) Each triangular region has a common side with at least one other triangular region.

When you get a polygonal region, draw its boundary on your paper. Then, try to find another polygonal region formed by the same rules that is not congruent to the first one. They should get:

These are the only three such non-congruent regions that meet the requirements. Stress that the regions are equivalent.

Exercises:

(1) Repeat the example problem using 5 such triangular regions.

(2) Repeat the example problem using 4 congruent square regions.

(3) Repeat the example problem using 5 congruent square regions.

Suggested extensions for those who find these intriguing:

- (a) use 6, 7, 8, ... triangular regions,
- (b) use 6, 7, 8, ... square regions,
- (c) use some number (3,4,5,...) of regions which are congruent regular polygons of more sides.

Challenge Problem:

Use 6 congruent square regions to form equivalent polygonal regions in the plane by following these rules:

(1) No square regions overlap.

(2) Each square region has a common side with at least one other square region.

(3) No four square regions have a point in common.

(4) There are no more than four square regions "in a row."

This is the challenge: Find all such non-congruent, equivalent polygonal regions (there are more than 20) and find a method of convincing yourself that there are no more.

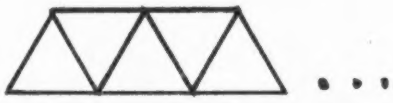
Note: There are 25 regions that fit the above rules (I think).

Bonus: Exactly 12 of your above regions can be cut out and folded along the common sides of the square regions to get a cube. Find them.

4.2 Decomposing regular polygons.

Establish the intuitive feeling that every regular polygon has a "center" and that by joining this center to each vertex, the regular polygon of n sides is decomposed into n congruent isosceles triangles.

Exercises: Give traceable patterns of regular polygons with the centers and the line segments joining each vertex to the center. Have students cut these out and arrange them in a line. That is,



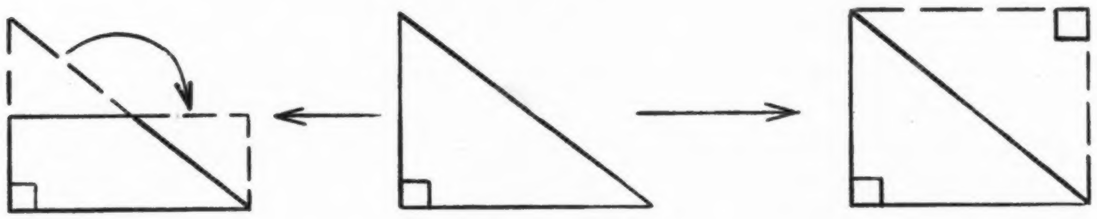
See if they recognize that the equivalent region so formed is always either a parallelogram or a trapezoid. See if they can generalize to a regular polygon of 71 or 72 sides.

4.3 Forming rectangular regions.

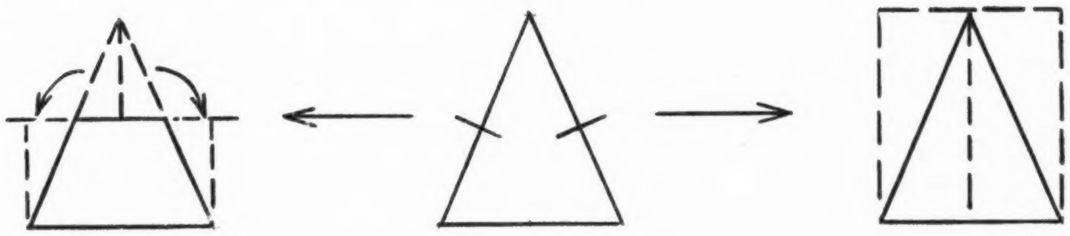
Discuss the "niceness" of measure as related to rectangular regions (remind them of approach to finding area; i.e., covering with square regions).

Exercises: Decomposing and doubling to get rectangles. Use scissors, encourage tracing regions, think freely.

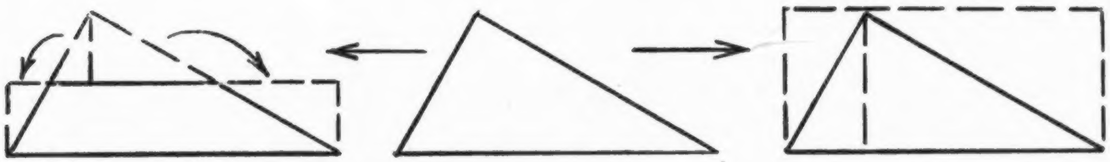
(1)



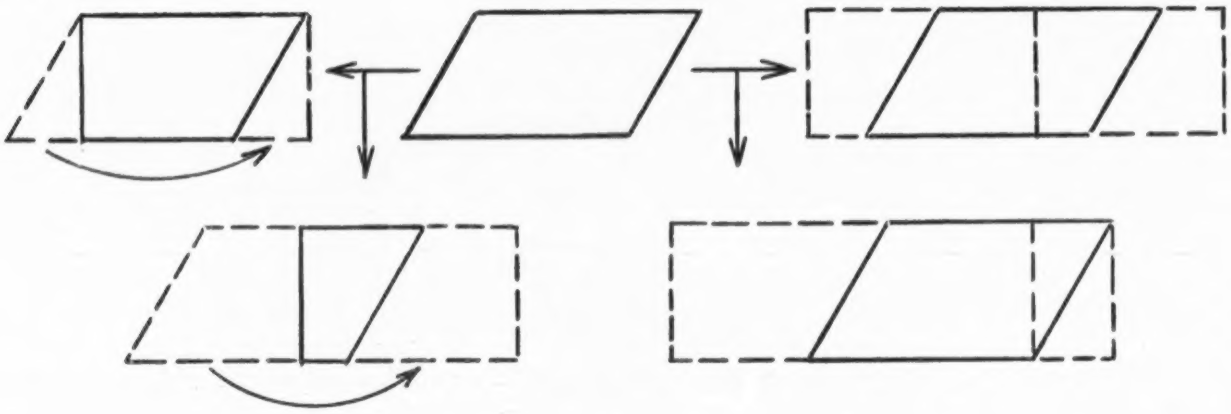
(2)



(3)



(4)



(5) Same ideas can be done with other familiar figures and with some that are not common such as a cross.

Note: Two possible Challenge Sections.

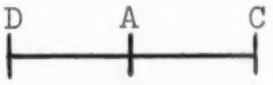
(1) Decomposing regions with various goals; e.g., any triangle to a parallelogram. Some of these could be extremely challenging and certainly non-trivial.

(2) Develop some of these same equivalence ideas for spatial regions.

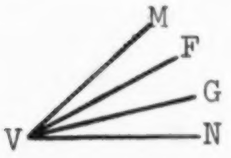
Section 5. Greater than (in a geometric sense) for segments, angles, planar regions, spatial regions.

Build situations where one set is a subset of another set of the same type. Some exploratory development leading toward the principle that the measure of a subset does not exceed the measure of the superset.

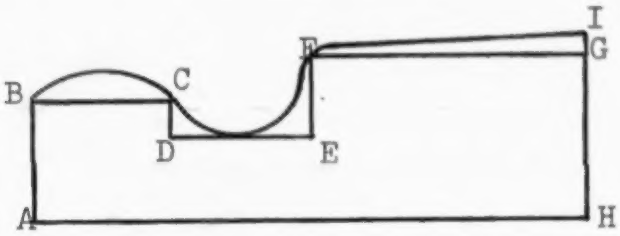
Exercises:



(1) Segment AD is a subset of CD.



(2) The interior of ∠FVG is a subset of the interior of ∠MVN.



(3) Polygonal region ABCDEFGHA is a subset of curvilinear region ABCFIHA.

(4) If a sphere is inscribed in a cube, the spherical region is a subset of the cubical region.

GRADE 7 - CHAPTER 6

RATIO AND SIMILARITY

Purpose:

To introduce the idea of not-necessarily-equivalent-but-metrically-related figures where one is a uniformly enlarged copy of the other--similar figures. To extend the idea of ratio and proportion. To relate the concept of geometrically similar figures to the more common usages in which the similarity may not be exact. To generalize isometry transformations to similarity transformations.

Rationale:

This chapter continues the process of structuring or modeling physical space begun in Chapter 1 (Nonmetrical Properties) and continued in Chapter 4 (Congruence and Replication) and uses the metric properties developed in Chapter 5 (Measure). Experience with physical objects suggests a generalization of congruence in which one figure is a magnified "copy" of another. This notion is idealized (modeled) in the concept of similar figures -- a figure which is a "uniform" enlargement of another. The general definition (barely hinted in the chapter) requires that the figures be related by a 1-1 correspondence which multiplies all distances by a fixed constant. Such correspondences, called similarity mappings, form a generalization of the concept of motion (or isometry).

Background Assumptions:

The concept of congruence; congruence of angles; some sufficiency theorems for congruence of triangles (SSS, SAS, ASA); the concept of linear measure; planar co-ordinate systems.

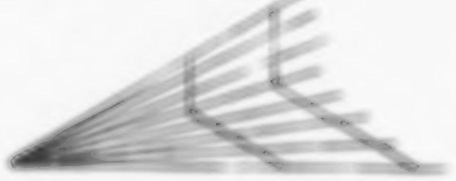
Section 1. Magnification and Contraction:

Note: This section is to be *exploratory* -- the physical examples and finger exercises are not to be developed or discussed at great depth. The geometric term "similarity" should not be used. The student should be allowed and encouraged to range widely within his experience for situations where some relatively pure magnification or contraction has occurred. Rely strongly on the students' intuitive belief that line segments will be transformed into line segments.

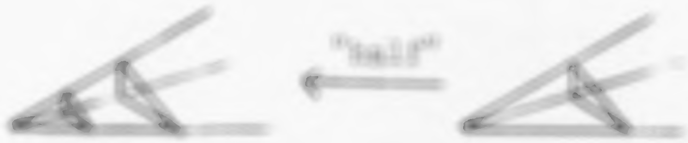
1.1 A discussion of blowing balloons (with pictures on them), projecting a slide (on a surface perpendicular to the projector), seeing through a telescope, microscope, and binoculars, taking a picture with a camera.

Exercises: A series of problems involving dilatation from a focal point by drawing and using a numerical property (e.g., directions "go twice as far" or "go half as far".) These should contain some pointwise dilatations for straights (and curves?) and some closed polygons. Students may be working in two- or three-space -- *don't ask*. A sample problem in detail occurs at the end of this outline (Appendix A).

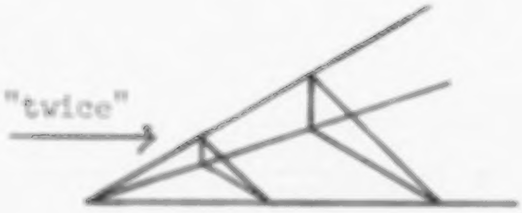
(1)



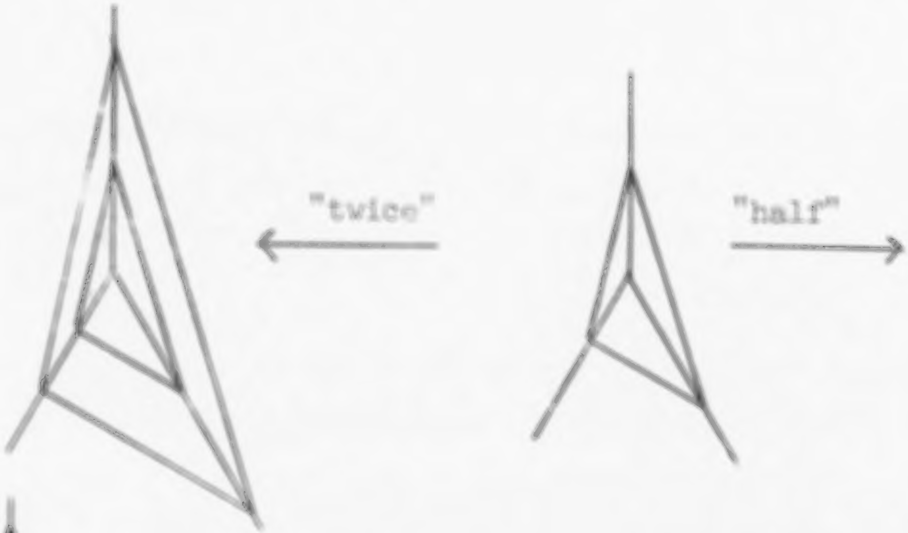
(2) (a)



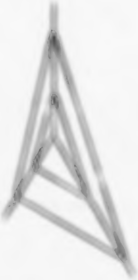
(b)



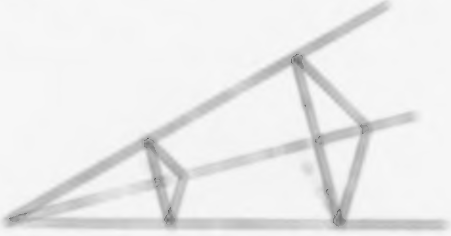
(3) (a)



(b)



(4) Might include a problem using focal point but no numerical property to show change in shape. For example, show a completed dilatation that was done incorrectly and have students explain what was done wrong and what happened differently as a result.

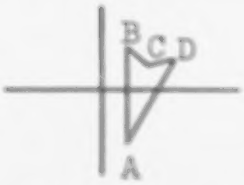


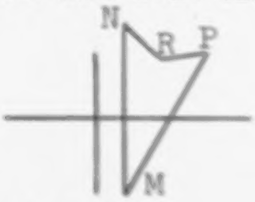
(5) Question for thought: do you see your pictures in these exercises as being in two-space or three-space?

1.2 A continuation of the above ideas by introducing a co-ordinate system to the plane and using the origin as the focal point. The written discussion might develop this very carefully for one problem. Nice example of adopting a model for neatness and clarity.

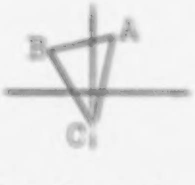
Exercises: A series of problems involving dilatation by use of co-ordinates and a numerical property (e.g., directions "double each co-ordinate" or "halve each co-ordinate"). Be sure to use some ratios such as $\frac{4}{3}$ for rational number practice (but only on positive numbers at this point). A sample problem occurs in detail at the end of this outline (Appendix B).

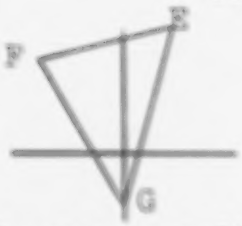
(1) A = (1,-1), B = (1,3) $\xrightarrow{\text{"double"}}$ M = (2,-2), N = (2,6)
C = (2,2), D = (4,2) R = (4,4), P = (8,4)
Plot and draw ABCD Plot and draw MNOP





(2) A = (1,3), B = (-2,2), C = (0,-1) $\xrightarrow{\text{"double"}}$ E = (2,6),
F = (-4,4)
G = (0,-2)
Plot and draw ABC Plot and draw EFG





(3) Might include a problem where only one co-ordinate is doubled to show change in shape. That is, double just the y co-ordinate and see the elongation.

(4) Might give two point sets where the points of one have been reflected in the origin and doubled to get the other. Have students graph each and see what happens. See if they can discern the manner in which the second point set was derived from the first.

(5) Check to see if the students recognize the focal points by having them draw rays from the origin through the vertices for a problem.

(6) Might have students draw some corresponding pairs of line segments in the similar figures and *try* to compare their relative positions and sizes.

Section 2. The Concept of Similarity:

Note: This section is still not definitive but is a more refined exploration.

2.1 Begin trying to refine the relationship of the figures worked with in the first two sections. See if students can recognize similar and non-similar figures and verbalize what they are looking for. Give some emphasis to what is not sufficient for this relationship.

Exercises:

(1) George's family moved into a new house and George told his friends: "My new bedroom is *similar* to the old one except that it is twice as big and has three windows instead of two". How is he using or misusing the geometric idea of similarity?

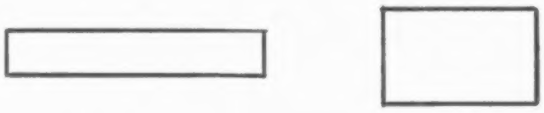
(2) Sarah said: "All people are *similar*: each has a head and two arms and two legs". How was she using the word "similar"?

(3) Is what a person sees with corrective eyeglasses *similar* to what he sees without his glasses?

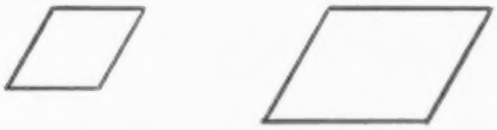
(4) A science teacher tells his class that a certain apparatus involving marbles and wires is *similar* to the solar system. Is it *similar* in the geometric sense of the word?

(5) Give some sets of figures where the student is to try to find similar figures and explain why he thinks they are similar. Make some "close" enough to get arguments. (Build the idea that we need something more than the eye or our rulers.)

(6) Compare any two rectangles o they necessarily have this similarity relationship we are looking for? (To build the idea that angle congruence is not enough.)



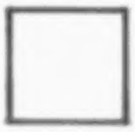
(7) Compare two equilateral quadrilaterals that are not congruent o they have this similarity relationship? (To build the idea that "doubling" sides is not enough.)



2.2 Further refining of this similarity relationship through a comparison to congruence (developed in Chapter 4) and equivalence (developed in Chapter 5). Develop a need for properties of ratio and proportion. Begin effort to get a definitive statement of this similarity relationship.

Exercises:

(1) Given a square



  (a) Make a congruent figure and explain why you think it is congruent.

  (b) Make an equivalent figure and explain why you think it is equivalent.

  (c) Make a similar figure and explain why you think it is similar.

(2) Repeat a, b, c for a triangle and some other shapes.

(3) Given: two triangles ABC and RST

$AB = 4$, $BC = 10$, $AC = 12$, $RS = 12$.

  (a) If these triangles are congruent, do you know the length of $\overline{ST}$? $\overline{RT}$?

(b) If these triangles are similar, do you know the length of $\overline{ST}$? $\overline{RT}$?

(4) Given: two triangles ABC and RST

$AB = 4$, $BC = 10$, $RS = 12$

(a) If these triangles are congruent, do you know the length of $\overline{AC}$? $\overline{ST}$? $\overline{RT}$?

(b) If these triangles are similar, do you know the length of $\overline{AC}$? $\overline{ST}$? $\overline{RT}$?

(5) Given: two triangles ABC and RST

$AB = 3$, $BC = 4$, $ST = 6$

(a) If these triangles are congruent, do you know the length of $\overline{AC}$? $\overline{ST}$? $\overline{RT}$?

(b) If these triangles are similar, do you know the length of $\overline{AC}$? $\overline{ST}$? $\overline{RT}$?

(6) Some exploratory problems relative to sufficiency situations; e.g.,

(a) If three angles of one triangle are congruent to three angles of another triangle, are the triangles similar? Why or why not?

(b) If four angles of one quadrilateral are congruent to four angles of another quadrilateral, are the quadrilaterals similar? Why or why not?

Section 3. Ratio and Proportion.

3.1 Develop the meaning of ratio and symbols for (rebuild from Grade 5, Chapter 9, but along the lines of MJHS Vol. 1, Chapter 9). Stress different names for the same ratio (see same references). Use some non-numerical ratios, such as line segments.

Exercises: see references.

3.2 Develop proportion as equality of ratios.

Property: $\frac{a}{b} = \frac{c}{d} \rightarrow ad = bc$

Property: $ad = bc \rightarrow \frac{a}{b} = \frac{c}{d}$ if $b \neq 0$, $d \neq 0$

Include a sequence of equal ratios and the idea of a constant of proportionality.

Example: $\frac{AB}{MN} = \frac{BC}{NO} = \frac{AC}{MO} = k$

If the measure of MN is 4, the measure of AB is ... (give a value for k in some problems but not all).

Exercises: See Grade 5, Chapter 9
MJHS Vol. 1, Chapter 9
Geometry, pp. 361-364

Section 4. Defining Similarity.

4.1 Formalize the definition of similarity for convex polygons (corresponding angles congruent and corresponding sides proportional).

Note: Do not stress the convex restriction. Broad definition is mainly to allow similarity of squares. Discuss problem of defining similarity for other figures (wiggly lines, simple closed curves). Stress correspondence property.

Exercises:

(1) Several problems where, given similar figures, the student sets up the correspondences.

(2) Some exploration of types of figures that are always similar: pairs of isosceles right triangles, equilateral triangles, 30-60-90 triangles, squares, (regular tetrahedrons, cubes?). Try to see why this is so and look for correspondences in more than one way.

(3) Discover the similarity of any two circles -- have students try to explain why.

(4) Problems that require completing the proportions for similar figures:

If $\triangle ABC \sim \triangle STR$ then $\frac{AC}{?} = \frac{CB}{?}$

Section 5. Sufficiency Properties for Triangles:

5.1 Exploratory work in class -- everybody draw a triangle (using portractors) with certain given angles, and compare with neighbors. Perhaps a written set of class exercises that will lead to the idea of AAA similarity.

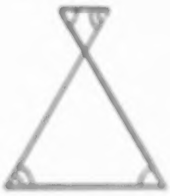
Short written discussion of sufficiency (relate back to congruence properties).

Statement of the AAA similarity property.

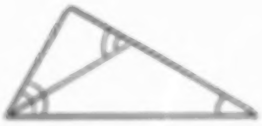
Exercises:

(1) Find similar triangles (a few short deductive sequences), and state the proportional sides, e.g.:.

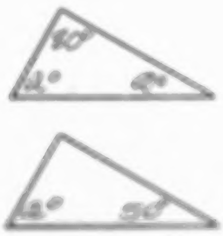
(a)



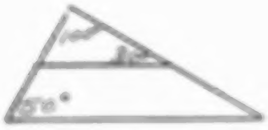
(b)



(c)



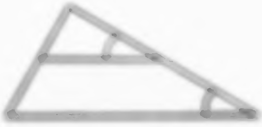
(d)



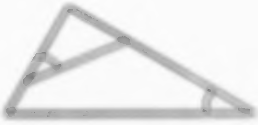
(2) Use the 180° property for triangles to develop AA similarity property.

(3) More finding similar triangles and stating proportional sides:

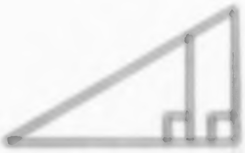
(a)



(b)



(c)



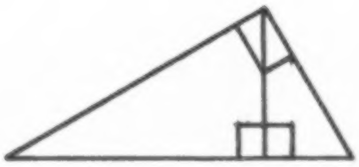
(4) What is sufficient for two right triangles to be similar?

(5) What is sufficient for two isosceles triangles to be similar?

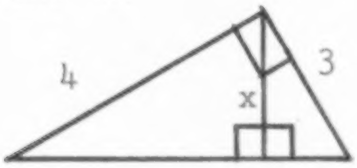
(6) Is there a congruence property like this similarity property? Why or why not?

(7) Brainbuster:

(a) Find similar triangles:



(b) Find x:



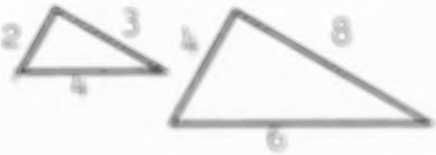
5.2 Exploratory exercises in class for SSS to be done by construction or with sticks (e.g., take a triangle -- form a new triangle by halving lengths). Again a set of written class problems might be useful.

Short rehash on sufficiency (perhaps SSSS for quadrilaterals). State SSS similarity property. Some more discussion on properties that contain three ratios rather than two.

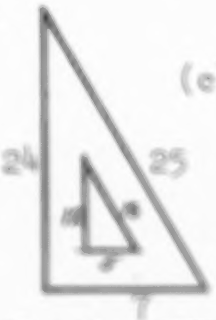
Exercises:

(1) Are these triangles similar? If so, state the correspondence of sides and angles.

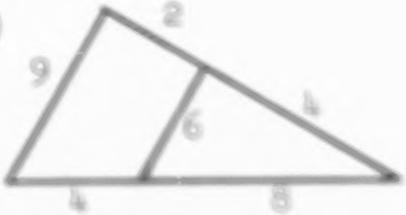
(a)



(b)



(c)



(2) Given triangle ABC and triangle RST, $\frac{AC}{ST} = \frac{BC}{RT} = \frac{AB}{RS}$. State the similar triangles showing correspondence.

(3) Given: AB = 5, AC = 3, CB = 7 and MN = 21, NO = 9, MO = 15. Are there similar triangles? Show why or why not.

(4) Why are all equilateral triangles similar?

(5) Is the proportionality of two pairs of sides of isosceles triangles sufficient for similarity?

(6) Is there a congruence property like this similarity property? Discuss why or why not.

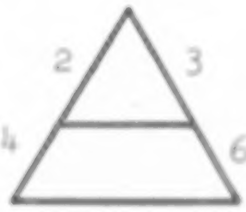
5.3 Exploratory Exercises in Class for SAS Similarity Property:

Statement of SAS property for similarity. Bring back proportion property $ad = bc \rightarrow \frac{a}{b} = \frac{c}{d}$ if $b \neq 0$, $d \neq 0$.

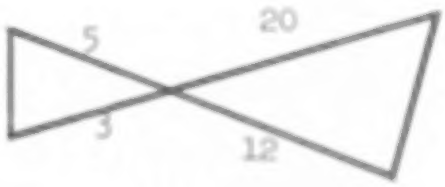
Exercises:

(1) Are these triangles similar? If so, state the correspondences involved.

(a)



(b)



(2) In triangles ABC and RST, $A \cong T$ and $AC \cdot TS = AB \cdot RT$. Are there similar triangles? If so, state the similarity. If not, discuss what is lacking.

(3) Does the line joining the midpoints of two sides of a triangle form a new triangle that is similar to the original triangle? Discuss why or why not.

(4) What is sufficient for two isosceles triangles to be similar?

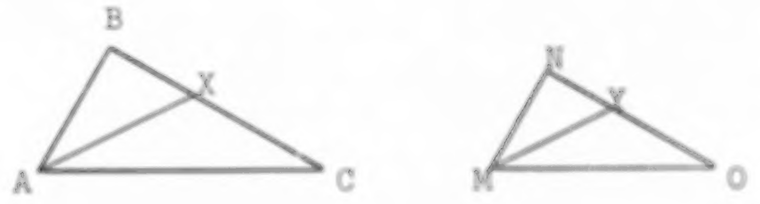
(5) Is there a congruence property like this similarity property? Discuss why or why not.

5.4 Add corresponding lines to two similar triangles (such as medians, altitudes, angle bisectors, midlines) and consider how they are related. Use this to review all three similarity properties and to reinforce the concept of similarity as being ratio-preserving for all corresponding linear parts.

(This section might be too difficult, but it is worth a try after they preceeding work.)

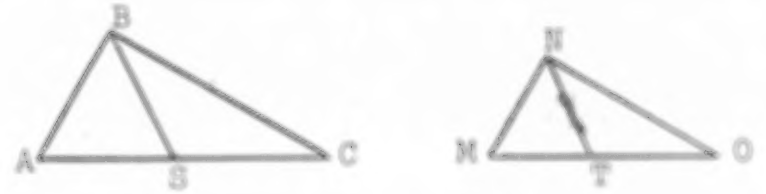
Exercises:

(1)



Given: $\triangle ABC$ - $\triangle MNO$, $\overline{AX}$ and $\overline{MY}$ are angle bisectors.
What can you deduce?

(2)



Given: $\triangle ABC$ - $\triangle MNO$, $\overline{BS}$ and $\overline{NT}$ medians.
What can you deduce?

(3) Might sneak in some questions relative to ratios of areas for two similar figures (such as squares).

Section 6. Similarity Mappings.

6.1 Find the similar triangles implied in Section 1.1. Go back to these and set up a mathematical model in which similar triangles can be found. Also use some old chestnuts like the height of the flagpole, the tennis serve, the river width, me and my shadow versus the tree (these are not "old" to them).

6.2 A thorough discussion of local maps as models for local geography (not stereographic maps, etc). A careful consideration of the limitations of the maps as models and what similarity (in the pure geometric sense) can be assumed. Thought should be given to hills and valleys, river paths, highway jogs, etc. and how these appear on the map. Teachers should be encouraged to bring in some local maps so the students can compare to an area they are "bicycle-familiar" with.

6.3 Working with scale drawing and blue prints (see NCHS, Vol. 2, pp. 384-389). Add some discussion of ratios of non-equivalent but comparable units (e.g., 1 foot to 1 mile is not 1:1 but 1:5280). Careful attention to how exact the "similarities" of blue prints and scale drawings are.

6.4 Natural place for some right angle trigonometry -- maybe just the tangent.

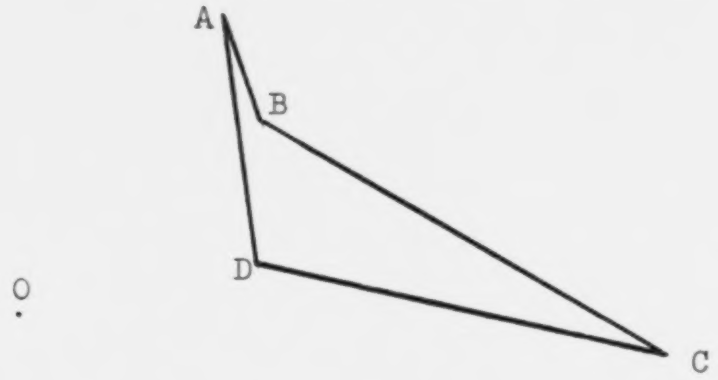
GRADE 7 - CHAPTER 6

APPENDIX A

1.1 Sample Problem in Detail:

Note: This problem would probably appear in the latter half of the problem set. However, figures might generally be non overlapping.

Given:



(a) Reproduce the given figure on your paper (by tracing if necessary).

(b) Draw $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OD}$, and $\overrightarrow{OC}$.

(c) Measure $\overline{OA}$ with a ruler. Multiply the length by $\frac{3}{2}$ and select a point R on $\overrightarrow{OA}$ such that $OR = \frac{3}{2} OA$.

(d) Measure $\overline{OB}$ with a ruler. Multiply the length by $\frac{3}{2}$ and select a point S on $\overrightarrow{OB}$ such that $OS = \frac{3}{2} OB$.

(e) Measure $\overline{OC}$ with a ruler. Multiply the length by $\frac{3}{2}$ and select a point T on $\overrightarrow{OC}$ such that $OT = \frac{3}{2} OC$.

(f) Measure $\overline{OD}$ with a ruler. Multiply the length by $\frac{3}{2}$ and select a point U on $\overrightarrow{OD}$ such that $OU = \frac{3}{2} OD$.

(g) Draw the polygon RSTU.

(h) Give some explanation of how ABCD and RSTU are related. (Note: not to be included in all problems.)

GRADE 7 - CHAPTER 6

APPENDIX B

1.2 Sample Problem in Detail:

Note: This would be a problem found in the latter half of the problem set. The exact terminology and symbols are dependent on preceeding development.

Given: A co-ordinate system with the point set $M = \{A,B,C,D,E\}$ where $A = (1,2)$, $B = (2,3)$, $C = (3,3)$, $D = (3,-2)$, $E = (1,0)$.

(a) Make a graph of M.

(b) Draw the polygon ABCDE.

(c) Form a second point set $N = \{R,S,T,U,W\}$ where $(x,y) \to (4x,4y)$ and $A \to R$, $B \to S$, $C \to T$, $D \to U$, $E \to W$. (That is, the co-ordinates of R,S,T,U,W are respectively four times those of A,B,C,D,E)

(d) Make a graph of N on the same co-ordinate system. (Note: In earlier problems, N might be graphed in a second co-ordinate system to avoid overlapping figures.)

(e) Draw the polygon RSTUW.

not to be included on all problems

(f) Give some explanation of how ABCDE and RSTUW are related.

(g) Draw $\overrightarrow{OR}$, $\overrightarrow{OS}$, $\overrightarrow{OT}$, $\overrightarrow{OU}$, and $\overrightarrow{OW}$ where O is the origin.

(h) What do you seem to observe about these rays?

GRADE 7 - CHAPTER 6

APPENDIX C

NOTE ON PERCENTAGE

After the concepts of ratio and proportion are developed in the early part of the chapter in connection with similarity, the following treatment may be used to develop a comprehensive view of percentage.

Since $38\% = .38 = \frac{38}{100}$ we may write a percent as a ratio. The following problems will illustrate how we may now work with ratios as percents.

Example 1: A movie theatre has 960 seats. If 15% of these seats are in the reserved section, how many seats are in the reserved section?

Let n represent the number of seats in the reserved section.

We know that the ratio of reserved seats to the total number of seats $= \frac{15}{100}$.

Also, the ratio of reserved seats to the total number of seats $= \frac{n}{960}$.

Since $\frac{15}{100}$ and $\frac{n}{960}$ are different names for the same ratio, then

$$\frac{15}{100} = \frac{n}{960} .$$

Example 2: The enrollment of a school is 950 students. Of these 228 are freshmen. What percent of the total school population is the number of freshmen?

Let n represent the number of percent of freshmen.

We know that $\frac{n}{100}$ represents the ratio of the number of freshmen to the total school population.

Also, the ratio of the number of freshmen to the total school population is $\frac{228}{950}$.

Since $\frac{n}{100}$ and $\frac{228}{950}$ are different names for the same ratio we have

$$n\% = \frac{n}{100} = \frac{228}{950} .$$

Example 3: A man found that he saved \$768 one year. If this was 8% of his total income for the year, what was his total income?

Let n represent the man's total income.

We know that $\frac{768}{n}$ represents the ratio of the savings to total income.

Also, $\frac{8}{100}$ represents the ratio of the savings to total income,

$$\frac{768}{n} = \frac{8}{100} .$$

The above three problem types include almost all significant problem types that the student is likely to encounter in percentage.

The percentage concept and its applications should be applied throughout.

Some applications will occur in (7-VII), (8-I), (8-IV), (8-VI).

GRADE 7 - CHAPTER 8

Background:

From Chapter 2: Opposite function. Opp: x → -x
Chapter 3: Addition and Subtraction of any two rationals, Percent
Chapter 6: Similar triangles

Objectives:

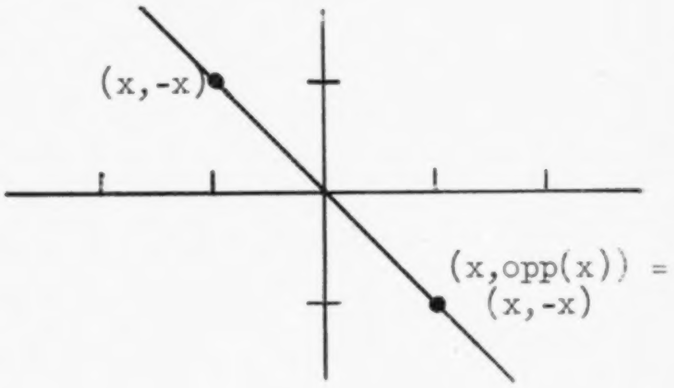
1. Multiplication of any two rationals.
2. The class of functions $f : x \to mx$ for all rational m.
3. Absolute value function.

Rationale: WST; hopefully this will be evident from the ensuing outline.

5.1 Review of negative rationals as a set of numbers. Opposite function and its graph

$$\text{Opp}(x) = -x$$
$$\text{Opp}: x \to -x$$

Observe that plotting a number of points on this function, we seem to obtain a straight line. Essay a proof using similar triangles.

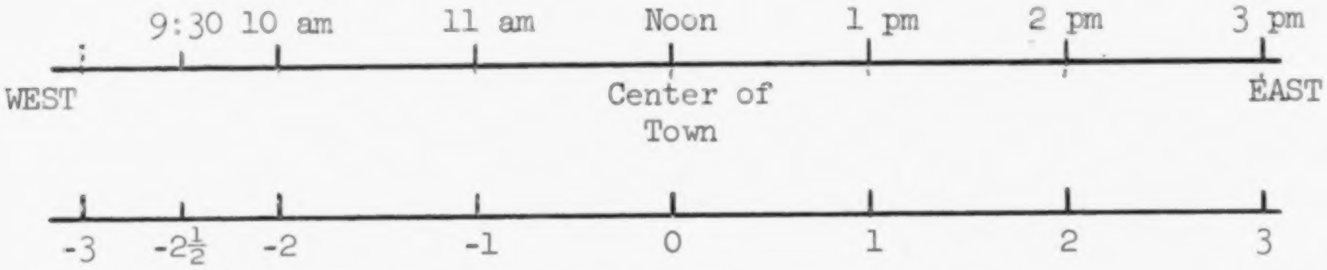


5.2 Multiplication of a positive rational by a negative rational. (For a similar but alternate treatment see Appendix to Chapter 3, Alternate version.)

A general rationale is exemplified by the following:

Let us consider the following reasonable situation: A man who lives on the west side of town starts walking eastward, along the main street of the town, at the rate of 2 miles per hour. At noon he reaches the very center of town. Natural questions are: Where will he be at 1 PM, at 2 PM, at 3 PM? Where was he at 11 AM, at 10 AM? If he started at 9:30 AM, where did he live?

A natural model, the number line suggests itself:

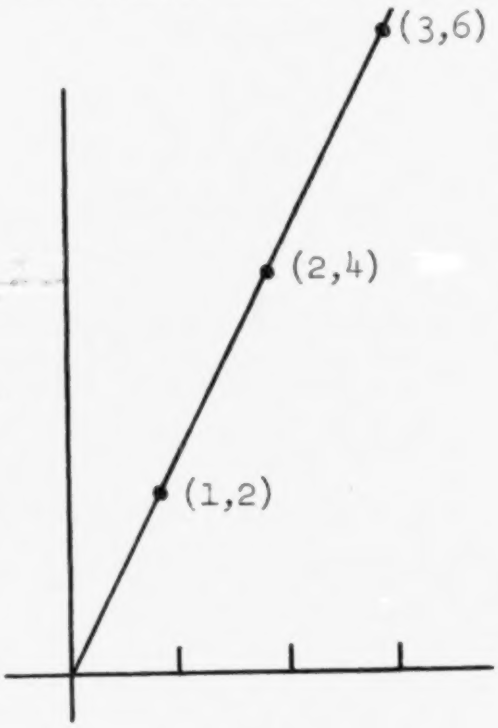


A simple application of the law that says:

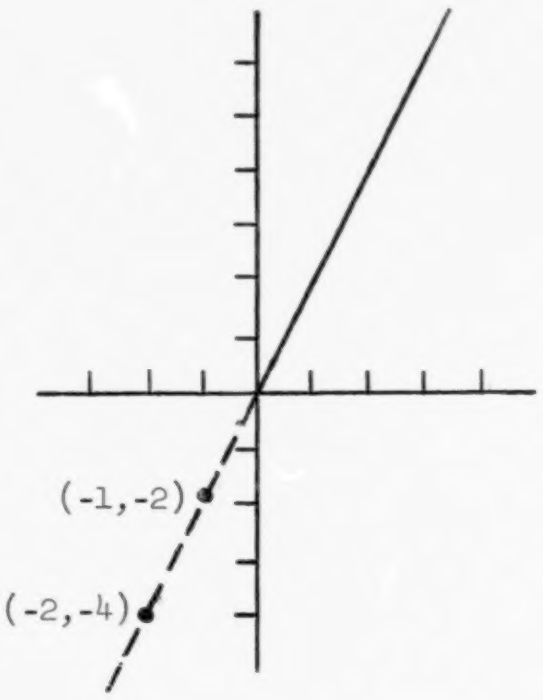
Distance traveled in miles in the time of T hours = Rate in miles per hour × Time in hours traveled, or more briefly, $D = R \cdot T$. In this case $D = 2T$ tells us that at 1 PM he is 2 miles east of the center of the town, at 2 PM he is 4 miles, and at 3 PM he is 6 miles east of the center of town.

Another model for way of representing our situation is to regard distance traveled as a function, $D : T \to 2T$, which expresses the association of time traveled with $2T$, the distance in miles traveled in T hours at the rate of 2 miles per hour.

Let us make a graph of this function. The origin corresponds to the man's location at noon the center of town. We shall plot several points. We



note an important result: The points appear to lie on a straight line! If we draw in the straight line, we may use it to read off other values. Suppose we wish to see how far our man has gone in, say, 10 minutes or $\frac{1}{6}$ hours. Our graph tells us: $\frac{1}{3}$ miles. Of course we can check this $\frac{1}{3} = 2 \times \frac{1}{6}$. The graph not only gives us a picture of how the distance traveled changes with time, but also permits us to obtain more information without further calculation.



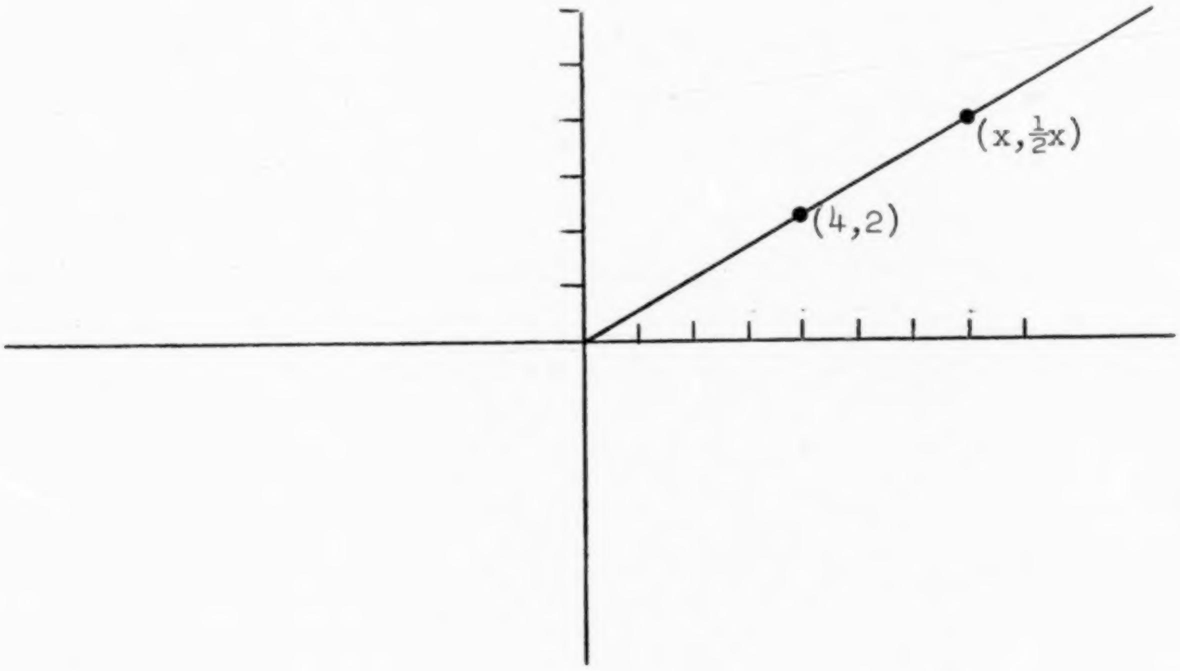
Now let us think about the question "Where was our man at 11 AM?" Surely he was west of the center of town, and indeed since 11 AM is 1 hour before noon, he was 2 miles west of the center of town. On the model which expresses our function $D : T \to 2T$ this suggests that the time 11 AM should be represented by (-1) (1 hour before noon) and the distance from the center of town is 2 miles west or -2. Thus we plot the point (-1,-2) to represent the information that 1 hour before noon he was 2 miles west of the center of town. In the same way the point (-2,-4) denotes the fact that 2 hours before noon (10 AM is represented by the negative number -2) our man was 4 miles west of the center of town (4 miles west is represented by the negative number -4). In the same way (-3,-6) would denote the fact that at 9 AM he was 6 miles west of the center of town. Of course, he began his walk at (9:30 AM, so (-3,-6) does not represent an attained position.

Now we observe that the points $(-2,-4)$ and $(-3,-6)$ lie on the extension of the line we have drawn. (An argument via similar triangles could be given here.) And so, by extending the line, we could obtain further information without further calculation. Thus we find that the point $(\frac{5}{2}, -5)$ lies on this line. The point $(-\frac{5}{2}, -5)$ corresponds to the fact that $\frac{5}{2}$ or $2\frac{1}{2}$ hours before noon (9:30 AM), our man was 5 miles west of the center of town. And, of course, we can check that, since in $2\frac{1}{2}$ hours he can walk $2\frac{1}{2} \times 2 = 5$ miles; thus he must have started 5 miles west of the center of town.

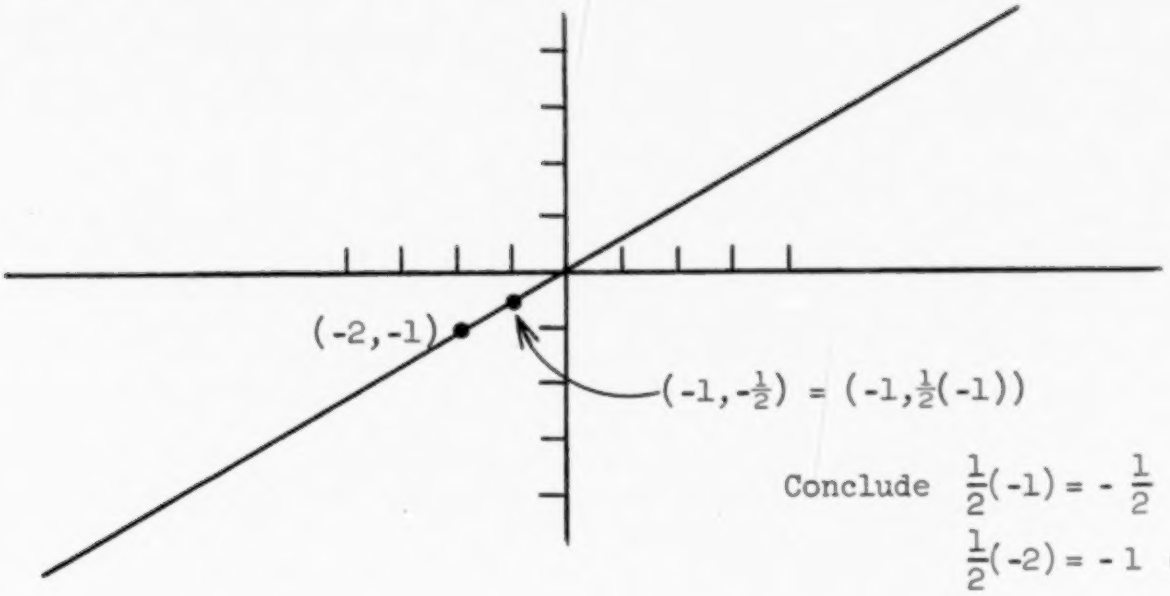
But now what of all this? By using negative numbers to represent times before noon and miles west of the center of town in our distance function $D : T \rightarrow 2T$, we conclude that $2(-1) = -2$ and $2(-2) = -4$ and $2(-\frac{5}{2}) = -5$.

Let us now consider some other examples of multiplication. Suppose the man had been crawling at $\frac{1}{2}$ miles per hour. What then? ... We plot several points. Again we find that they lie on a straight line.

"Carry On"



Extend to



Conclude $\frac{1}{2}(-1) = -\frac{1}{2}$

$\frac{1}{2}(-2) = -1$ etc.

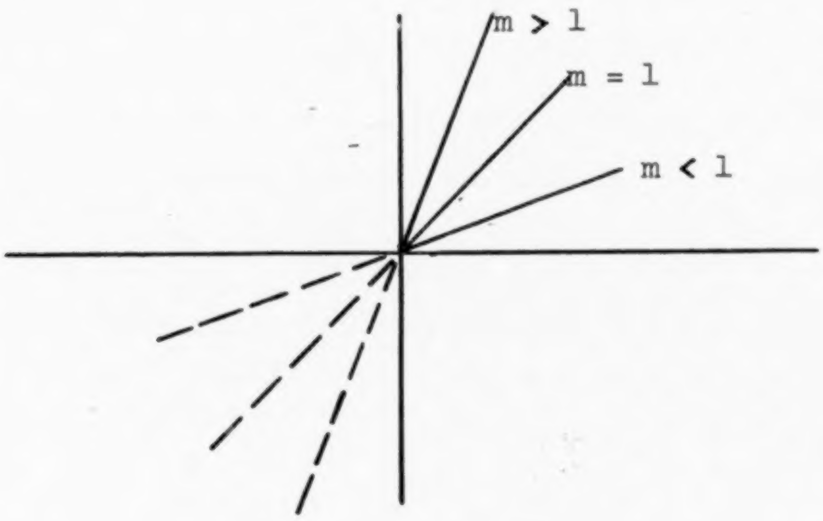
5.3 Graphs of multiplication by a positive rational number.

From the $D = RT$ example go to multiplication by 3, $\frac{1}{2}$, $\frac{7}{5}$, and generalize to multiplication by $m(m \geq 0)$. First do for mx for positive x and then extend line into Third Quadrant. Plot lots of points and obtain a little drill in the multiplication of rationals. Denote numbers both by fractions and by decimals.

Multiplication by $m(m > 1)$

by $1(m = 1)$ (the identity function)

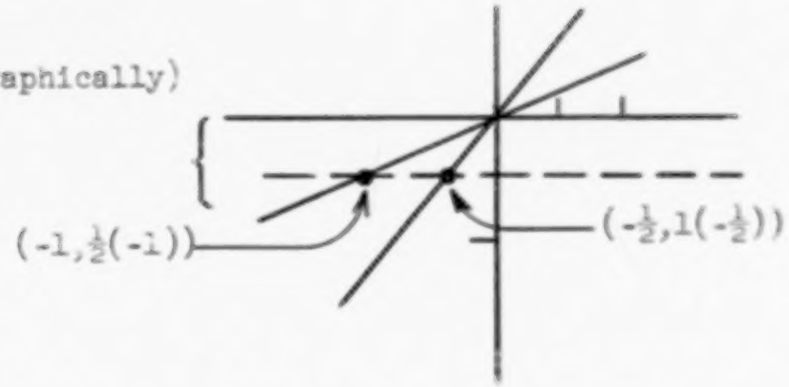
by $m(m < 1)$



Extension into Third Quadrant.

$a(-b) = b(-a)$ (do graphically)

$$-\frac{1}{2} = 1(-\frac{1}{2})$$
$$= \frac{1}{2}(-1)$$



Obtain rule: If $a > 0$ and $b > 0$ then $a(-b) = -(ab) = b(-a)$.

5.4 Multiplication of a positive by a negative and the distributive law.

From the distributive law: Since $ab \rightarrow a(-b) = a(b + (-b)) = a \cdot 0$
$$= 0$$

Then $a(-b) = -(ab)$

Some space should be given here to the mathematical philosophy of what laws a system of numbers should obey and the fact that the rules we have for multiplication are merely definitions, albeit ones which were suggested in a natural way.

5.5 Multiplication by a negative rational.

Now that we have multiplication of a positive by a negative we can consider the function $f : x \rightarrow (-2)x$ when x is positive. Plot points of this form: $(x, -2x)$ when x is positive. This leads to a line in the Fourth Quadrant. Many of the steps in the previous sections can be repeated.

Point out that the opposite function is also multiplication by -1. Extend line for $x \rightarrow -2x$ into Fourth Quadrant.

Obtain Rule: If $a \geq 0$ and $b \geq 0$ then $(-a)(-b) = ab$.

Use distributive law to show agreement of this result. Point out that this derivation yields rule:

For all rationals, $(-a)(-b) = ab$ and $(-a)b = -(ab)$. Check all cases with the previously graphed functions. (A discussion of how many cases will arise is an interesting combinatorial problem.)

Establish commutativity of multiplication for the rationals.

5.6 Addition and subtraction revisited. Use $-a = (-1)a$.

Example:

$$-2 - 3 = -2 + (-3) = (-1)2 + (-1)3 = (-1)(2 + 3) = (-1)5 = -5 .$$

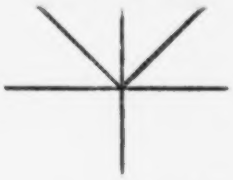
5.7 More on Opposite function

$$-(-2,-3) = (-1)(-2,-3) = (-1)(-5) = 5$$

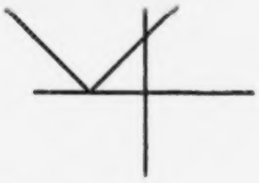
Opp. $(x - 1)$; its graph

5.8 Absolute value function; their graphs.

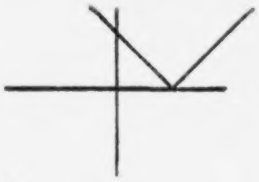
$x \rightarrow |x|$



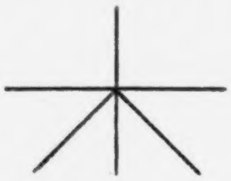
$x \rightarrow |x + 1|$



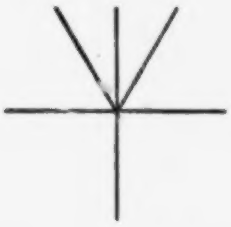
$x \rightarrow |x - 1|$



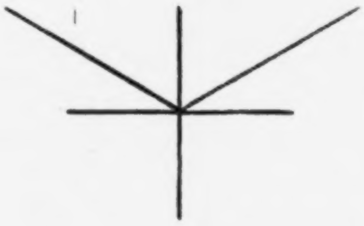
$x \rightarrow -|x|$



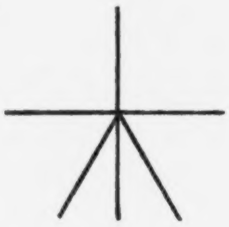
$x \rightarrow |2x| = |-2x| = 2|x|$



$x \rightarrow |\frac{1}{2} x|$



$x \rightarrow -2|x|$



5.9 Applications.

Any you can think of, but certainly percent decrease problems.

5.10 Graphing $x \rightarrow ax + b$; the role of the parameters a and b.

Discuss the cases:

(1) $a = 0$

(2) $a = 1, \ b > 0$

(3) $a = 1, \ b < 0$

(4) $a > 0, \ b > 0$

(5) $a > 0$, $b < 0$

(6) $a < 0$, $b > 0$

(7) $a < 0$, $b < 0$

Slope and intercept.

GRADE 7 - CHAPTER 8

(Alternate Version)

(To accompany "Alternate Versions" of Chapters 3, $3\frac{1}{2}$)

GRAPHS OF LINEAR FUNCTIONS; VARIATION

Background:

1. Addition, subtraction, multiplication and division of rational of rational numbers.
2. An introduction to "percent" function, the opposite function, the absolute value function.
3. Coordinates, graphs of linear functions in the first quadrant.
4. Solutions of mathematical sentences.
5. Ratio and Similarity

Rationale:

This chapter will extend the concepts of functions and graphing presented in Chapter 2. The operations with rational numbers will be available from Chapter 3 along with beginning techniques for solution of mathematical sentences. Ratio and Similarity also provide a realistic background for the discussion of slope and concurrency.

Purpose:

1. To provide some understanding of and some skill in graphing functions of the form $f : x \rightarrow mx + b$.
2. To provide a strong graphical background for the solutions of systems of sentences by Linear Combination.
3. To provide a background of Variation as a function, graphical representation and applications.

Section 1. Graphs of Functions:

1.1 Review briefly the rectangular coordinate system and association of points with their coordinates.

1.2 Graphs of Functions with restricted domains.

Example 1: Graph the following four functions on the same coordinate axes:

(1) $S : x \rightarrow 2x, \quad 0 \leq x \leq 2$

(2) $M : x \rightarrow 2 , \quad 2 \leq x \leq 4$

(3) $O : x \rightarrow 2x - 8, \quad 4 \leq x \leq 6$

(4) $G : x \rightarrow 2x - 14, \quad 7 \leq x \leq 9$

Example 2: Graph the following function:

$T : s \rightarrow s^2, \quad 0 < s$ (tie in Area Function)

Example 3: Graph the following function:

$$K : x \rightarrow \begin{cases} -1, & 0 \leq x < 2 \\ 1, & -2 \leq x \leq 0 \\ 2, & 4 \leq x \leq 6 \end{cases}$$

Example 4: Graph the following function:

$$A : x \rightarrow \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

Section 2. Slope and Intercepts:

2.1 Review graphing of linear functions in first quadrant.

2.2 Experimentally develop concept of slope.

Example: Draw the graphs of the following functions on the same coordinate axes.

$$S : x \rightarrow 3x$$
$$L : x \rightarrow 3x + 1$$

$$O : x \to 3x - 3$$
$$P : x \to 3x - 4$$
$$E : x \to 3x + 5$$

Ask students to select two arbitrary points on each line (probably a lattice points to start with) and then write the ratio of the difference of the y coordinates to the difference of the x coordinates for each pair of points. (Select additional points and repeat process). Have students identify the coordinates of the points where these lines cross the x and y-axes.

Example: Continue the same process with several sets of functions $f : x \to mx + b$ where m and b assume many different rational values.

2.3 From the students work in Section 2.2 abstract a simple notion of slope in terms of the change in the y and x coordinates of points on the graph of a linear function and the notion of the y-intercept.

2.4 Discuss slopes of linear functions $f : y \to a$ and lines parallel to the vertical axis. (WST) Discuss slopes of parallel lines and intersecting lines.

Prove: 3 points are collinear if the slopes of segments determined by them are the same.

Section 3. A Closer Look at Slope.

3.1 Discuss the cases $f : x \to mx + b$

(1) $m = 0$

(2) $m = 1$ and $m = -1$, $b > 0$

(3) $m > 0$ where $b > 0$

(4) $m < 0$ where $b > 0$

(5) $m = 1$ and $m = -1$ where $b < 0$

(6) $m > 0$ where $b < 0$

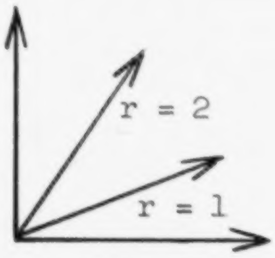
(7) $m < 0$ where $b < 0$

3.2 Discuss (WST) the idea of increasing and decreasing linear functions.

3.3 Discuss, in particular, $f : x \rightarrow mx$ in terms of m being a "multiplier" as in the function for distance, rate and time

$$d : t \rightarrow rt .$$

How the graph of this function changes when r is doubled, halved, increased by 2, etc., can be investigated.



Another example:
Consider the function $A : w \rightarrow \ell w$

Section 4. Variation: (See MFJHS, Vol. 2, Part 2, pp. 392-411)

4.1 Direct variation $d : x \rightarrow kx$ with applications and graphs.

4.2 Inverse variation $I : x \rightarrow \frac{k}{x}$

4.3 Other kinds of variation $S : x \rightarrow Kx^2, \ldots$

Section 5. Discuss solutions of equations like:

$3x + 2 = 5x - 3$ by graphing

$f : x \rightarrow 3x + 2$

$g : x \rightarrow 5x - 3$

Section 6. Scale drawings as a function with applications.
(See ref. above.)

$$S : x \rightarrow \frac{1}{1000} x \text{ , etc.}$$

(Tie in with Chapters on Measure, Rates and Similarity).

GRADE 7 - CHAPTER 8

(Alternate version)

APPENDIX

Subtitle: A simple minded motivated approach to the multiplication of negative numbers.

We wish to define multiplication for all rational numbers. When this is done, and their basic properties investigated we shall have developed a mathematical system which is sufficiently strong to solve a wide variety of useful problems. In earlier sections we have seen that we could add and subtract any two rational numbers to good advantage, although the rules for doing so were rather complicated. We may expect that the rules for multiplication are also complicated. It is a pleasant relief to learn that they are not quite as bad as the ones for addition. One might feel that if mathematicians were worth their salt they would devise ways of doing these arithmetic operations more easily, but it turns out that if we want to define addition and multiplication so that the nice properties we found for the positive numbers are satisfied there is really only one way to do this. This important theorem we shall not prove here. Rather, we shall spend our time trying to show the naturalness of our definitions. There is no need for secrecy, as far as the definitions themselves go. We shall define multiplication so that

$$(-a)b = -(ab) = a(-b) \quad \text{and} \quad (-a)(-b) = ab.$$

Thus $(-2)3 = -6$, $(-3)2 = -6$ and $(-2)(-3) = 6$.

Fortunately these definitions can be interpreted physically; indeed the need for these mathematical manipulations arises in all sorts of mathematical applications and we should want our definitions to reflect this. We shall consider an example, albeit a somewhat artificial one in an attempt to give a natural motivation for our mathematical definition.

Example 1: The Weather Bureau reports that it is now 0 degrees Fahrenheit ($0^\circ F$) and that for several hours the temperature has been steadily rising at the rate of 2° per hour. They forecast that this rise will continue throughout the day. Assuming this forecast to be correct, a number of natural questions arise: What will the temperature be one hour from now, 2 hours from now, 3 hours from now? What was the temperature one hour before now, 2 hours before now? If the minimum temperature occurred $2\frac{1}{2}$ hours before now, what was it?

The answers to these questions are intuitively easy: One hour from now the temperature will be 2° above zero, 2 hours from now the temperature will be 4° above zero and 3 hours from now the temperature will be 6° above zero. How about before now? Since the temperature is rising, it must have been colder one hour ago. Indeed, since the change in the temperature is a rise of 2° an hour, it must be that one hour before now the temperature was 2° below zero, or -2°. Similarly, two hours ago the temperature was -4° and $2\frac{1}{2}$ hours ago it was -5°. Let us represent this data in a table:

Time	Temperature in degrees
Now	0
1 hour from now	2
2 hours from now	4
3 hours from now	6
1 hour before now	-2
2 hours before now	-4
$2\frac{1}{2}$ hrs. before now	-5

It is clear that the top half of our table is constructed by multiplying the number of hours from now by the rate at which the temperature is increasing. In symbols: $\text{Temp} = 2 \cdot \text{Time}$, or in a more abbreviated form, if we let F stand for the temperature in degrees Fahrenheit and T for the time passed in hours we have a formula: $F = 2T$.

The important observation to make in this: We can still use this rule for determining the bottom half of the table if we make two other interpretations. To begin with, let us use a negative number to represent time before now. Thus we denote, one hour ago as -1, 2 hours before now we denote as -2, and so on. Remember, the temperature is rising at the rate of 2° per hour and we have interpreted this as a positive 2. Thus if the rule $F = 2T$ is to hold here it must be that $2(-1) = -2$. That is we must define

$$2(-1) = -2$$

and similarly that $2(-2) = -4$ and $2(-2\frac{1}{2}) = -5$. More generally we will want to define

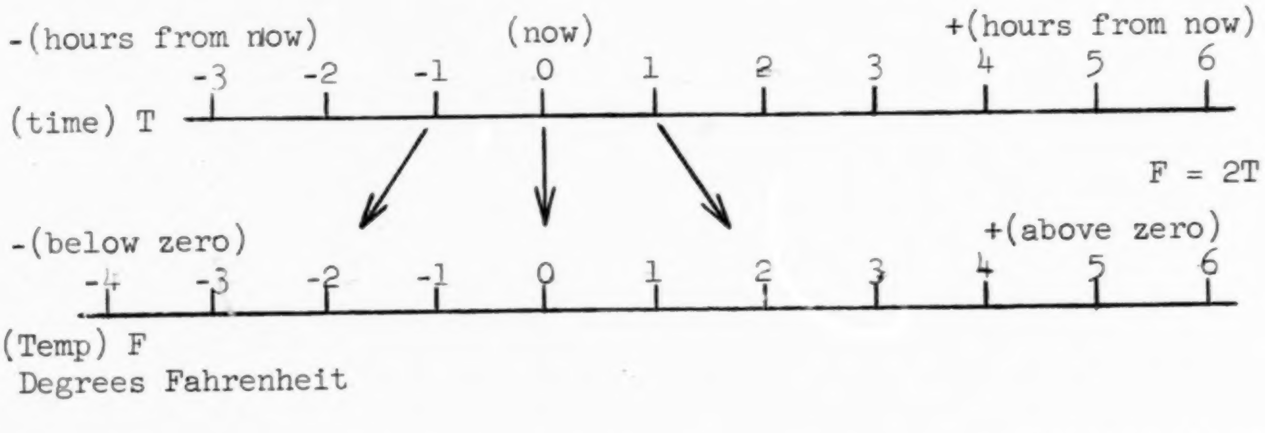
$$\text{If } b > 0 \text{ then } 2(-b) = -(2b).$$

Under this definition the formula $F = 2T$ holds for all of the table.

The rule $F = 2T$ is an indication that once again, a function is at work in the background. The function is the association of time, T, with the temperature at time T.

$$T \rightarrow 2T$$

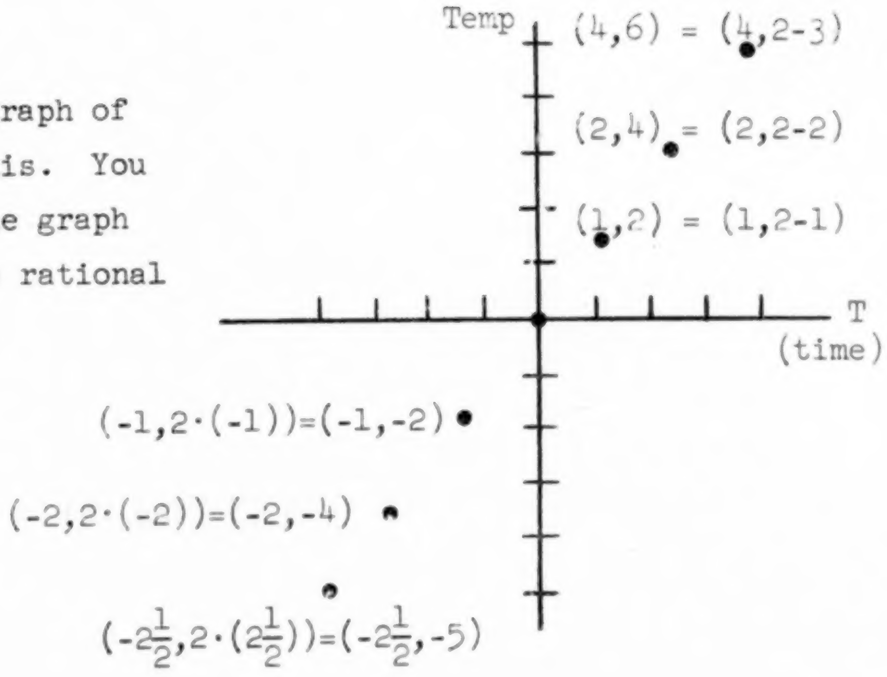
The association works as follows:


-(hours from now)
(now)
+(hours from now)
-3
-2
-1
0
1
2
3
4
5
6
(time) T
F = 2T
-(below zero)
+(above zero)
-4
-3
-2
-1
0
1
2
3
4
5
6
(Temp) F
Degrees Fahrenheit

Draw in some more arrows to show how the numbers are associated by the function.

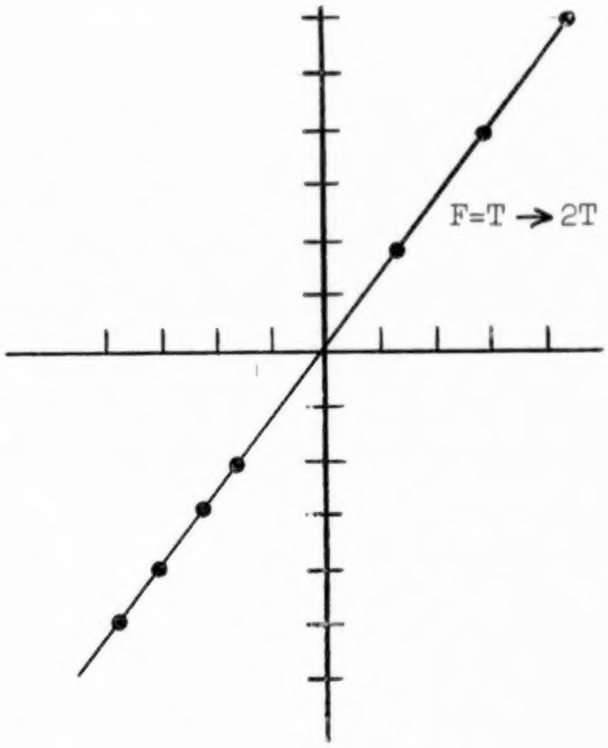
If we plot the pairs (T,2T) the graph of the function T → 2T looks like this. You should plot some other points of the graph of this function, especially choose rational values for T between 0 and 4.

T	2T
1/2	1
3/4	3/2
2/3	4/3
5/4	5/2



This graph gives us in essence a picture of multiplication by 2. It is important to note that in the first quadrant the points lie on a line.

Let us draw in the suggested line. If we extend this line into the third quadrant we find that the points (-1,-2), (-2,-4), $(-2\frac{1}{2},-5)$ lie on this line. Thus, once more, we agree that the line should picture multiplication by 2, that is, the points on the whole line should be of the form (T,2T) and we are forced to define



$$2(-1) = -2, \text{ etc.,}$$

and in general

$$\text{If } b > 0 \text{ then } 2(-b) = -(2b).$$

In the rule $2(-b) = -(2b)$ what happens if we let b equal -1? $(-b)$ becomes $(-(-1)) = 1$. And so $2(-b)$ equals 2. Now what about $-(2b)$ if b equals -1? Is it true that $2 = -[2(-1)]$? Now

$2(-1) = -2$ by what we have just learned, and so the question becomes, does $2 = -(-2)$? This answer is of course YES. Thus, by very similar arguments for the general case, $2(-b) = -(2b)$ for all b, positive, zero, or negative.

Not let us consider a second and colder example:

Example 2: The Weather Bureau reports that it is now 0° F and that the temperature is steadily falling (decreasing) at the rate of 2° F an hour.

Again we can ask the "natural" questions: What will the temperature be one hour from now, 2 hours from now, 3 hours from now; what was it one hour before now, 2 hours before now, $2\frac{1}{2}$ hours before now?

Again the answers are easy to determine and we list our results in a table just as before.

Time	Temperature in degrees	
Now	0	
1 hour from now	-2	(2 degrees below zero).
2 hours from now	-4	
3 hours from now	-6	
1 hour before now	2	(2 degrees above zero)
2 hours before now	4	
$2\frac{1}{2}$ hrs. before now	5	

Our arguments for these calculations are similar to the ones in Example 1 except that in this problem it is getting colder, temperature decreases. Can we use the rule $F = 2T$? NO! -- And we shouldn't expect that rule to work since "+2" was used to represent an increase in temperature. Therefore it is natural to use (-2) to denote a decreasing rate of changing temperature. Thus our rule should now become

$$F = (-2)T.$$

If this is our rule, then it tells us that one hour from now the relation $-2 = (-2)1$ should hold. The rule should give us for the relation holding two hours from now that $-4 = (-2)\cdot 2$, and 3 hours from now that $-6 = (-2)3$. In general, a hours from now $-(2a) = (-2)a$. Thus we are led to the general rule for multiplication

$$\text{If } b > 0 \text{ then } (-2)b = -(2b).$$

Now what about the rule $F = (-2)T = -(2T)$ for the bottom half of the table? One hour before now the temperature was 2° above zero $(+2)$. Again we must interpret time before now as a negative number; one hour before now is represented by -1. And if our rule $F = (-2)T$ is to hold for this case it must be that

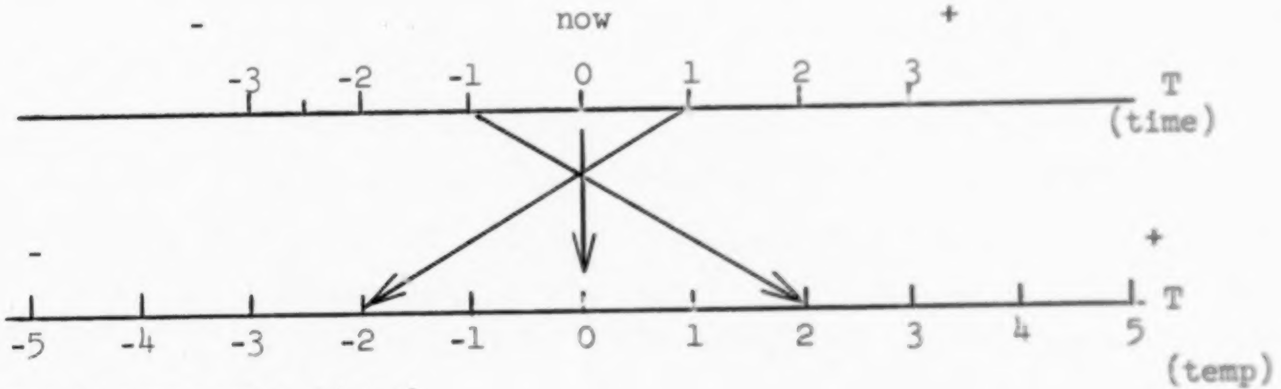
$$2 = (-2)(-1).$$

If we make this definition then the rule $F = (-2)T$ will hold. Similarly, we define $(-2)(-2) = 4$ and $(-2)(-2\frac{1}{2}) = 5$. In general we have

$$\text{If } b > 0 \text{ then } (-2)(-b) = 2b.$$

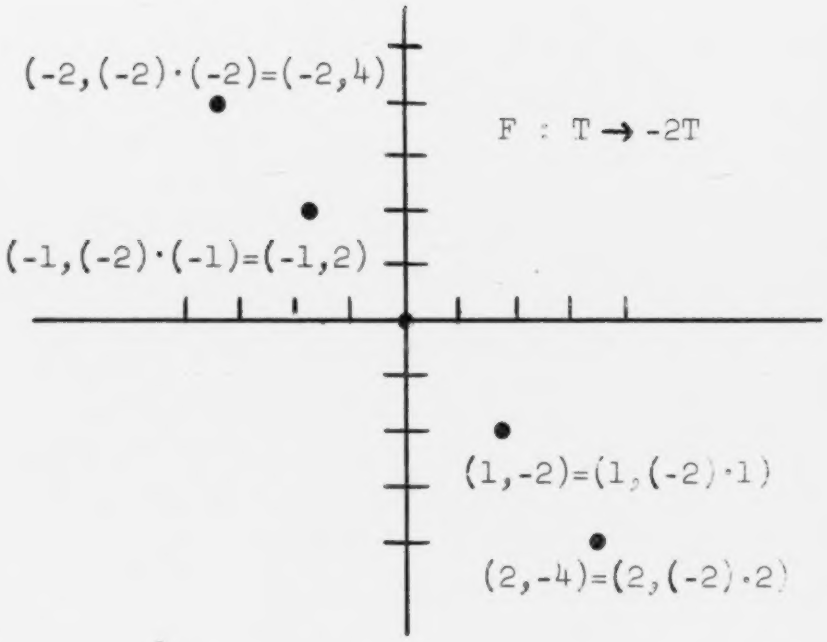
With this definition the rule $F = (-2)T$ holds for the entire table given in Example 2.

Again this rule means that a function is at work. The function associates $T \rightarrow (-2)T$ and it can be displayed as follows:

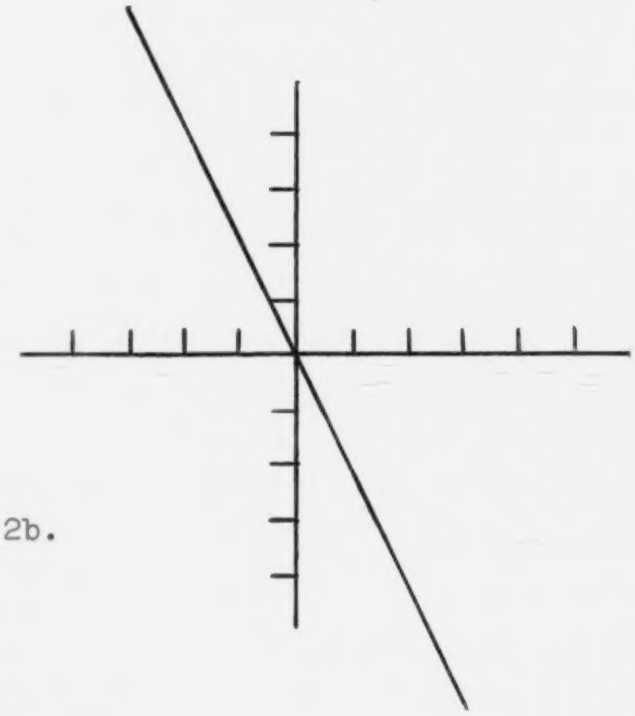


Draw in some more lines!

As before let us plot some points corresponding to the entries in the table. Again they appear to lie on a straight line in the next figure. The points corresponding to entries from the top half of the table appear in the fourth quadrant; the points corresponding to entries from the bottom half of the table appear in the second quadrant.



This graph gives us picture of multiplying numbers by (-2). Again we see that it is natural to define $(-2)(-1) = 2$, $(-2)(-2) = 4$, and so on.



In general:

If $b > 0$ then $(-2)(-b) = 2b$.

Our rules thus far are:

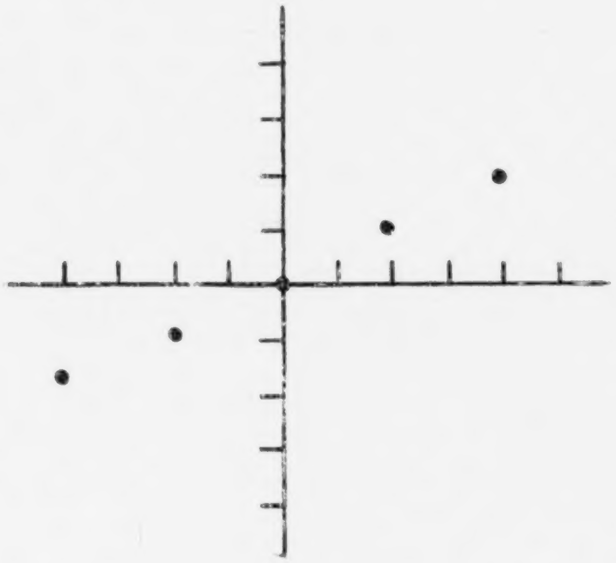
1. $2(-b) = -(2b)$
2. $(-2)b = -(2b)$ if $b > 0$
3. $(-2)(-b) = 2b$ if $b > 0$.

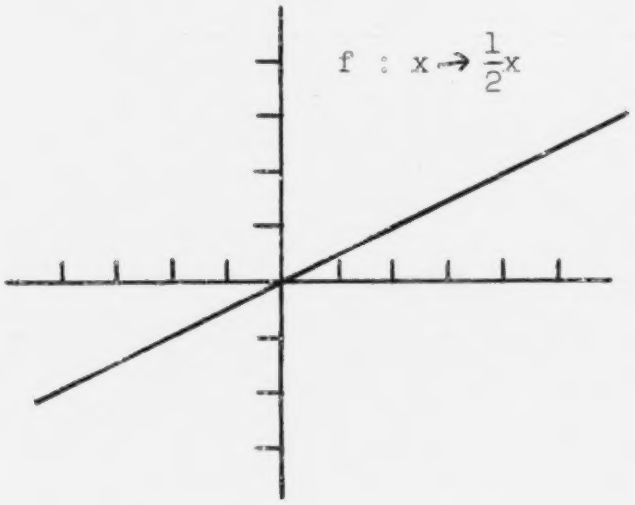
The rules 1 and 2 resemble each other and speaking loosely, they say that we can slide a "minus sign" from one factor to another, or take it "outside" the parenthesis.

We can in fact show that 2 and 3 hold for all b, positive or negative -- or zero. For example, in 3 if we try using a negative number for b, say $b = -3$, (and thus $-b = 3$) we get $(-2)(3) = 2(-3)$ and thus by 1 we can conclude that $(2)(3) = -(2 \cdot 3)$. But this is just the result of 2 when $b = 3$. Thus we can see that 3 holds for $b = -3$, and similarly we can see that 3 holds for all b. In just the same way we can show that 2 holds for all b.

Now it must be easy to see that in either of our examples, if we had changed the rate of temperature rise or fall the resulting rules for multiplication would have been the same. That is, the actual value "2" had nothing to do with the real part of our reasoning. However it is still informative to sketch the development from the function and graphing point of view, and to see the effect of multiplication by various numbers. For example, let us replace "2" by "$\frac{1}{2}$". Let us consider the function $x \to (\frac{1}{2})x$.

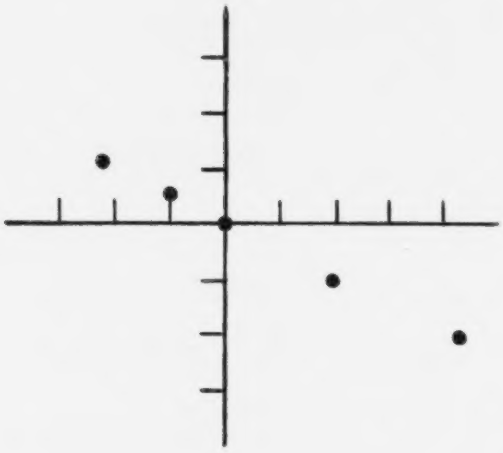
Plot some points in the first quadrant and then draw the line, extending it into the third quadrant. Now plot some points which should fall into the third quadrant. Do they fall on the line? Now read some point from the line. If your point is, say, (c,d) is $d = (\frac{1}{2})c$?

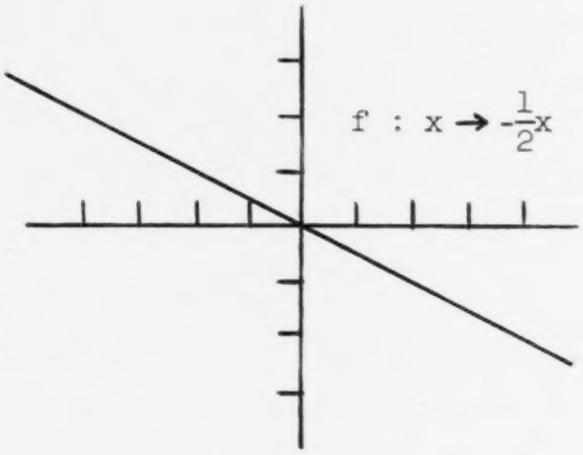




Compare the graphs of the functions $x \to 2x$, $x \to x$, and $x \to (\frac{1}{2})x$. It may be helpful to plot all three graphs on the same figure.

Now consider the function $x \to -(\frac{1}{2})x$. Plot points and see whether they seem to fall on a line. They should! Carry out the same investigations as with the previous example.





When you are done compare the graphs of the functions

$$x \to -2x \qquad x \to -x \qquad \text{and} \qquad x \to -(\tfrac{1}{2})x .$$

A general query: If $a > 1$, where will the graph of $x \to ax$ lie in comparison with $x \to x$. What if $0 < a < 1$?

If $a < -1$ compare the graphs of $x \to ax$ and $x \to -x$.
If $-1 < a < 0$ compare the graphs of $x \to ax$ and $x \to -x$.

From these examples and their obvious extensions to all rational numbers we obtain the following rules as the basis of multiplication on the set of all rational number.

1. $a(-b) = -(ab)$
2. $(-a)b = -(ab)$
3. $(-a)(-b) = ab$

Another possible explanation of these rules lies in an application of the distributive law. If the distributive law is to hold for our numbers then we can argue as follows:

$$0 = 2 \cdot 0 = 2(1 + (-1)) = 2 \cdot 1 + 2(-1) = 2 + 2(-1).$$

Since $0 = 2 + 2(-1)$ it is clear that we must have $2(-1) = -2$. Similarly, since $0 = 0 \cdot 1 = (2 + (-2)) \cdot 1 = 2 \cdot 1 + (-2) \cdot 1 = 2 + (-2) \cdot 1$ it is clear that $(-2) \cdot 1 = -2$. Knowing these two results we have

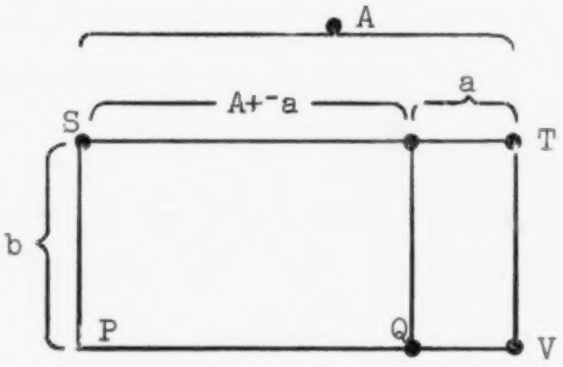
$$0 = -2 \cdot 0 = (-2)(1 + (-1)) = (-2) \cdot 1 + (-2)(-1).$$

Now since $(-2) \cdot 1 = -2$ we have that $0 = -2 + (-2)(-1)$. Thus $(-2)(-1) = 2$.

We can of course perform these arguments more generally by replacing 2 by a and 1 by b to obtain derivations of the three rules above from the distributive law applied to negative numbers.

Perhaps a more important observation for us is that similar calculations provide a proof of the distributive law for all rational numbers, if we have defined our multiplication already by the three rules above.

Finally there is a nice interpretation of these laws in terms of area. The area of rectangle PQRS is $b(A + (-a))$. On the other hand it is



Area of PVTS - Area of QVTR

or

$$bA - ba.$$

Hence $b(A + (-a)) = bA - ba$. But from distributivity again we have

$$b(A + (-a)) = bA + b(-a).$$

Hence $b(-a) = -(ba)$.

GRADE 7 - CHAPTER 9

SOLUTIONS OF SYSTEMS OF EQUATIONS AND INEQUALITIES

Background Assumptions:

1. Rational numbers with the four fundamental operations.
2. Graphing of linear functions.
3. Familiarity with terms half-line, half-plane, intersection of lines, etc.
4. Solution of simple equations and simple verbal problems.
5. Set language, including "union" and "intersection".

Rationale:

Introduce a problem for which it is convenient to use two variables and write two equations, as motivation for the solution of systems of equations. Graphic solution is developed, then a need is created for an algebraic solution to deal with problems whose solutions are not integral. For this purpose, the "comparison" method is introduced.

Systems leading to parallel or to coincident lines will be considered. Graphs of simple inequalities will be discussed, as a background for later work in linear programming.

Purpose:

1. To extend the student's knowledge of graphs to enable him to solve certain types of problems.
2. To show how algebraic methods can sometimes be of help in getting a more exact result than is possible through graphs.

3. To consider intuitively systems for which the solution set is either $\emptyset$ or an infinite set, thus leading to parallelism in a later chapter.

4. To develop better understanding of simple inequalities through graphing their solution sets in the coordinate plane.

5. To reinforce concepts of union and intersection of sets.

Procedure:

Section 1. Solving systems of equations:

1.1 Introduce a simple problem which is more easily solved by writing two equations than by confining oneself to a single equation, e.g.,

Problem: Mary bought 3 notebooks and 5 pencils and paid 70 cents; John bought 2 notebooks and 10 pencils, of the same kinds, and paid 80 cents. What was the price of one notebook?

Point out that if notebooks cost x cents each, and pencils cost y cents each, we can write the two sentences

$$3x + 5y = 70$$
$$2x + 10y = 80 .$$

Now we need to discuss what this means in terms of the solution being an ordered pair that satisfies both equations; i.e., the intersection of the solution sets.

1.2 To graph the equations and thus find the intersections of the graphs, it is convenient to write the "y-form" of each -- i.e., in each case we write a rule for a function which assigns to each x a value for y which gives a point (x,y) on the line. Thus we have:

$$y = -\frac{3}{5}x + 14$$

$$y = -\frac{1}{5}x + 8 .$$

1.3 We graph the two equations (see 9H, pp. 789-792) and find that they intersect at (15,5). Thus the price of one notebook is 15 cents.

1.4 Here point out that, since at the intersection the ordered pair is the same for each equation, we could have bypassed the graph and moved from the two equations in 1.2 to the equation in one variable

$$-\frac{3}{5}x + 14 = -\frac{1}{5}x + 8 .$$

Solving algebraically leads us to the solution 15, and thus to 15 cents as the price of a notebook.

1.5 Following some practice on equations with integral solutions, lead into ones whose solutions are not integral, to emphasize the need for the algebraic method.

Section 2. Systems which do not have unique solutions.

2.1 Have a student graph such a system as

$$\begin{cases} y = 3x + 2 \\ y = 3x - 5 \end{cases}$$

Look for intersection and observe none. Students will probably see that the lines have same slope -- promise discussion of parallel lines in a later chapter.

2.2 Look at algebraic solution of the system above:

$$3x + 2 = 3x - 5 .$$

Note that its solution set is $\emptyset$.

2.3 Now give him one like

$$\begin{cases} 2x + 3y = 6 \\ 4x + 6y = 12 \end{cases}$$

to graph. In putting into the y-form, he gets the same equation for both, hence only one line. Here slopes and y-intercept are correspondingly equal.

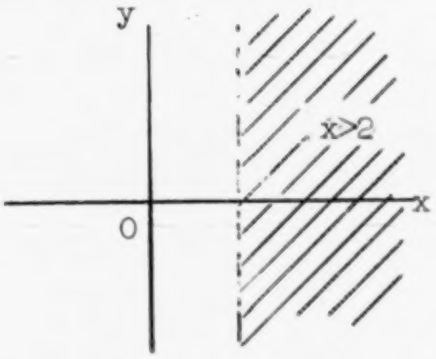
2.4 Point out algebraic solution gives equation

$$-\frac{2}{3}x + 2 = -\frac{2}{3}x + 2$$

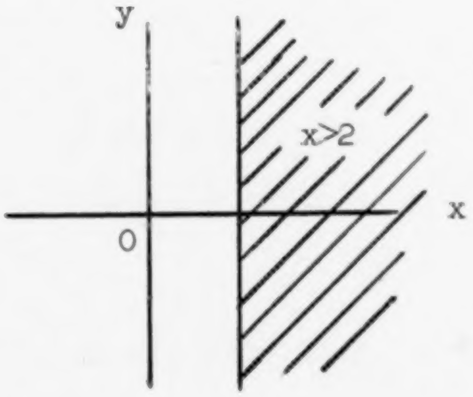
which is true for all values of x. Solution set is line $y = -\frac{2}{3}x + 2$.

Section 3. Graphs of Inequalities: (F. C., pp. 418-421; 9H, pp. 752-760)

3.1 Discuss graph of $x > 2$ as answer to question "where are all of the points which represent ordered pairs such that the first coordinate is greater than 2?" Here point out that a half-plane is involved, and discuss how to indicate it by shading, and that the line $x = 2$ should be a dash line to indicate that it is not included.

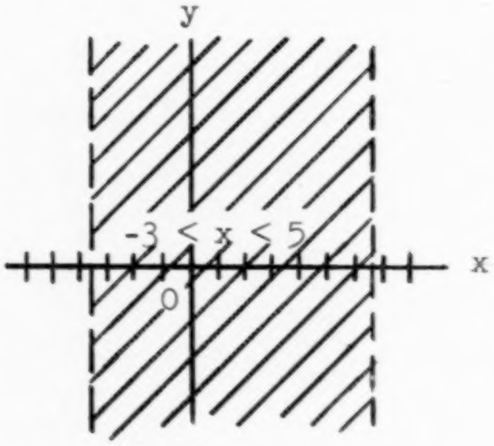


3.2 Show that the graph of $x \geq 2$ is the union of the half-plane $x > 2$ and the line $x = 2$. Then provide practice on others of this sort - such as $y < 3$, $y \geq 4$, $x < 8$, etc.



3.3 What about graph of $y > x$, $y < x$, $y \geq 2x$?, etc.? Discuss these in terms of half-planes with edge $y = x$, $y = 2x$, etc.

3.4 Consider graphs which are "strips" - e.g., $-3 < x < 5$.



(Note: if above notation has not been used before, here is the place to talk about it as abbreviation for:

$-3 < x$ and $x < 5$, or for
$x > -3$ and $x < 5$.)

3.5 Graphs of inequalities involving absolute value. Compare graphs in plane of

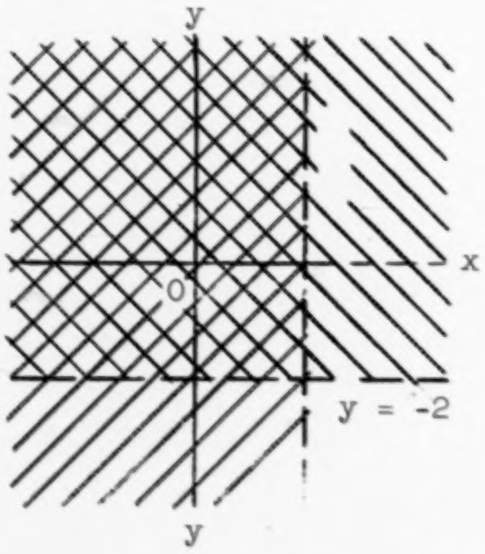
$|x| = 3$,
$|x| > 3$,
and $|x| < 3$.

Section 4. Systems of Inequalities: (9H, pp. 820-821)

4.1 Discuss intersection of half-planes, etc.

$$\begin{cases} x < 2 \\ y > -2. \end{cases}$$

Point out "doubly shaded" area as the graph of "$x < 2$ and $y > -2$".



4.2 Consider other systems, such as

$$\begin{cases} y < x \\ x > -2 \end{cases}$$

and

$$\begin{cases} x < 5 \\ -3 \leq y \leq 5 \end{cases} .$$

GRADE 7 - CHAPTER 10

DECIMALS; SQUARE ROOTS; REAL NUMBER LINE

Background:

1. Expanded notation for decimal, using integral exponents.
2. A rational number $\frac{a}{b}$ can be expressed as a repeating (or terminating) infinite decimal; approximation of a rational number to specified number of digits.
3. Points for rational numbers on the number line.
4. Pythagorean Theorem.
5. Unique factorization property.
6. Properties of the rational number system, especially density.
7. Operations with rational numbers, using decimal notation.

Note: While this background is assumed, review will surely be called for.

Purpose:

To develop belief in the existence of irrational numbers, using geometric and arithmetic approaches.

To develop ability to approximate square roots.

To examine the properties of the real number system.

Procedure:

Section 1. Motivation for irrational numbers:

1.1 Recall of familiar sets of numbers: counting numbers, whole numbers, non-negative rationals, integers, rationals. Properties of each set shared with previous sets, properties not shared with previous sets.

1.2 Use of Pythagoream Theorem to find length of one side of a right triangle, given the lengths of the other two sides.

(a) All lengths counting numbers: 3, 4, 5; 6, 8, 10; 5, 12, 13; etc.

Use of unique factorization property to find square roots of numbers which are perfect squares: $\sqrt{576}$, $\sqrt{225}$.

(b) All lengths rational: $\frac{3}{2}$, 2, $\frac{5}{2}$; 2, $\frac{15}{4}$, $\frac{17}{4}$.

$\sqrt{\frac{a}{b}}$, where a and b are perfect squares.

(c) One length irrational: 3, 2, $\sqrt{13}$.

What about $\sqrt{13}$? It is the length of a segment, but what kind of number is it? A counting number? A rational number?

Section 2. Intuitive argument for the theorem: If n is a counting number and $\sqrt{n}$ is a rational number, then $\sqrt{n}$ is an integer.

Exercises similar to those in 1.2(c), in which square roots are to be described by inequalities; e.g., $\sqrt{5}$ is a number between 2 and 3 which is not a rational number.

Section 3. Decimals which name rational numbers:

3.1 Recall of $\frac{a}{b}$ definition for rational number.

Use of division algorithm to find decimal name; argument that the decimal will have a repeating block of no more that b - 1 digits. (ISSM Vol. 1, p. 525.)

3.2 Given a repeating decimal, find the $\frac{a}{b}$ name for the number. (ISSM, Vol. 2, p. 366, MJHS, Vol. 2, p. 250.)

3.3 Generalization that every repeating decimal names a rational number.

3.4 Repeating decimals in which the repeating block is 0 (terminating).

(a) Review of expanded notation.

$$\begin{aligned} 0.567\overline{0} &= 0 + 5\left(\frac{1}{10}\right) + 6\left(\frac{1}{10}\right)^2 + 7\left(\frac{1}{10}\right)^3 + 0 \\ &= 0 + 567\left(\frac{1}{10}\right)^3 + 0 \\ &= \frac{567}{(2 \cdot 5)^3} \\ &= \frac{567}{(2^3 \cdot 5^3)} \end{aligned}$$

(b) Theorem: The decimal for $\frac{a}{b}$ (if a and b are relatively prime) will have a repeating block 0 if b has only 2's or 5's or both as prime factors.

Use of theorem to determine whether decimal for a given $\frac{a}{b}$ will terminate, and, if so, the number of digits before the repeating block 0 begins. (MJHS Vol. 2, p. 251.)

Section 4. Decimals which name irrational numbers:

4.1 Possibility of designing patterns for infinite decimals which do not repeat. Such decimals are names of irrational numbers. .1010010001... (ISSM Vol. 2, p. 370.)

4.2 Recall from Section 2 that square roots of most counting numbers are not rational, i.e., irrational.

Procedure for approximating square roots of counting numbers by iteration method. (JHSM Vol. 2, p. 262.)

Question: Should a flow chart be attempted?

Section 5. The real number line:

5.1 Location of points for rational numbers.

(a) Using fraction name: separate each unit segment into the number of congruent parts named by the denominator and count off the number of those parts indicated by the numerator.

(b) Using decimal name: show successively smaller intervals which contain the point.

$.\overline{51}$ is on the segment with endpoints $.5$ and $.6$
$.51$ and $.52$
$.515$ and $.516$ etc.

Include such cases as $.\overline{9}$.

5.2 Location of points for irrational numbers.

(a) Location of points for square roots of counting numbers.

Use diagonals of rectangles with vertex at the origin to locate points for $\sqrt{2}$, $\sqrt{3}$, $\sqrt{4}$, $\sqrt{5}$, etc.

Use additive property of segments on a line to locate points for such numbers as $(\sqrt{2} + \sqrt{3})$ and $3\sqrt{5}$. Also locate points for negatives. (ISSM Vol. 2, p. 357, MJHS Vol. 2, p. 257.)

(b) Location of points for non-repeating infinite decimals: as in Section 5.1(b), show successively smaller intervals which contain the point. (ISSM Vol. 2, p. 360.)

5.3 Real number line as union of set of points corresponding to rational numbers and set of points corresponding to irrational numbers.

Section 6. Density of the real numbers:

6.1 Density of rational numbers.[1]

Name a rational number between $\frac{2}{3}$ and $\frac{3}{4}$.

Name a rational number between $.\overline{1}$ and $.\overline{10}$.

6.2 Density of irrational numbers.

Name an irrational number between .101001000... and .101101110...

Name an irrational number between $\sqrt{2}$ and $\sqrt{3}$.

6.3 Density of real numbers.

Name a rational number between .101001000... and .101101110...

Name an irrational number between $.\overline{1}$ and $.\overline{10}$.

GRADE 7 - CHAPTER 10

APPENDIX

WHAT, ANOTHER PROOF THAT $\sqrt{2}$ IS IRRATIONAL?

If there is any novelty in this proof it is that it does not appear to use the unique factorization theorem. We give a proof for the special case $\sqrt{2}$ which does not appear to have a generalization to $\sqrt{n}$, even if n is not a perfect square!

The proof does assume a familiarity with odd and even integers and these facts:

1. The product of two odd integers is an odd integer.
2. The product of two integers, one of which is even is even.

In fact, if this proof should be presented to 7th graders, it might be well to make a table of all cases:

A	B	A × B
odd	odd	odd
even	odd	even
odd	even	even
even	even	even

And now the proof:

Let us suppose that we could discover a rational number $\frac{A}{B}$ such that $\frac{A}{B} = \sqrt{2}$. This would, of course, mean that $\frac{A^2}{B^2} = \frac{2}{1}$, or $A^2 = 2B^2$. Since A and B are integers, we may ask two simple questions: Is A even or odd? Is B even or odd?

First we ask, can A be odd? If this were possible, then A^2 would be odd and so $A^2 = 2B^2$ states that an odd number is equal to an even number which is impossible. Thus, our first conclusion is

If $A/B = \sqrt{2}$ then A is even.

Now what about B? Can B be odd? If this were possible then B^2 is odd and we have $A^2 = 2B^2$. Now, remember we now know that A is even; let us say $A = 2a$ and so $A^2 = 4a^2 = 2B^2$ and so

$$2a^2 = B^2.$$

Again this states that an odd number (B^2) equals an even number $2a^2$. Hence we have our second conclusion, which we combine with our first conclusion:

If $A/B = \sqrt{2}$ then A is even and B is even.

Finally, we are ready for the coup de grace!

Can $\sqrt{2}$ be rational? We claim not! Suppose we could, from all the fractions $\frac{x}{y}$ such that $\frac{x}{y} = \sqrt{2}$, select the one in which x is the smallest possible positive integer. Suppose we write that fraction as $\frac{A}{B}$. We are now in deep trouble because we have assumed

$$\frac{A}{B} = \sqrt{2}$$

and yet our conclusion above states that A and B are both even. Since this is so, we may divide both A and B by 2; say $\frac{A}{2} = a$ and $\frac{B}{2} = b$. Thus, $\frac{A}{B} = \frac{a}{b}$ and so $\frac{a}{b} = \sqrt{2}$ and yet a is a smaller positive integer than A. But this contradicts our choice of A!

Thus, any assumption that $\sqrt{2}$ is rational inescapably leads us to a contradiction. Our only consistent conclusion is that there can be no rational number whose square is 2.

GRADE 7 - CHAPTER 11

PARALLELISM

Background:

Many of the ideas of parallelism have been discussed prior to this chapter, and even more of them are rather intuitively simple for students. But there is a need to summarize, every so often, ideas that are scattered about in previous intuitions and in previous logical presentations. This is the first of such summaries which concerns itself with parallelism. Each future summary will, of course, extend and deepen these ideas.

We are assuming some understanding of points, lines, and planes in 2 and 3 dimensions. Such ideas as intersection and intersecting as set and relation respectively are needed; also the ideas of incidence: point is on line, line contains point, point is in plane, plane contains point, line is in plane, plane contains line.

There must have been some discussion of 2 dimensional graphs and of linear equations, but not much use is made of these in this chapter. We will need the meaning of polygon, regular polygon, side, angle of polygon, tetrahedron, edge, face and vertex of tetrahedron, and measure of segment, angle, area and volume.

Purpose:

The purposes of the chapter are as follows:

1. To strengthen, and sometimes to define, the concepts of parallel and skew relationships in 2 and 3 dimensions.
2. To illustrate the concepts of parallelism for the basic 1 and 2 dimensional figures.

3. To associate 2 and 3 dimensional concepts as often and as fully as possible at this level of mathematics.

4. To interconnect concepts of similarity with those of parallelism in simple ways as related to parallel lines and planes.

5. To connect synthetic concepts of parallelism with analytic geometry for lines parallel to the coordinate axes only.

6. To study the transversal figures in 2 and 3 dimensions noting especially the relationships between congruent angles and parallel lines.

7. To develop slightly some ideas of proof with respect to parallelograms and rhombuses and to use such an approach to review congruent triangles.

8. To consider parallel nets of lines and planes.

Rationale:

Parallelism at this point of the curriculum, followed by perpendicularity (at the beginning of Grade 8), summarizes two of the fundamental ideas of geometry, affine and metric. These chapters illustrate how many, individual ideas come together into larger, more general and more abstract concepts.

Also, these concepts are needed to help clarify such later concepts as linear equations in two dimensions, and much later in three dimensions; the concept of translation which relates to both transformation and vector; and to locus problems, trigonometry and complex numbers. Once parallelism is introduced we can present formally such figures as parallelograms, rhombuses and trapezoids and the theorems which concern the measures of their sides and angles and of their areas. Parallel planes permit discussion of prisms and later of cylinders.

Section 1. Parallel one-dimensional objects:

1.1 Line parallel to a line defined as two lines in the same plane which do not intersect.

1.2 Skew lines introduced to reinforce the need for "in a plane" being in the definition.

1.3 Extension of parallelism to rays and segments; use idea of "carrying line" or, if you prefer, "line containing segment", "line containing ray".

1.4 Network or grid of equidistant parallel lines in plane; use both perpendicular sets and non-perpendicular sets; also use exercises in which the units are the same on the two axes and other exercises in which the units are not the same.

Typical Exercises:

1. Draw the following figures:

   (a) In 2-space, two lines parallel to a fixed line. Can these two new lines intersect in just one point?

   (b) In 3-space, two lines parallel to a fixed line. Can these two lines intersect in just one point?

   (c) In 2-space, two lines each parallel to a given line through a point not on that line.

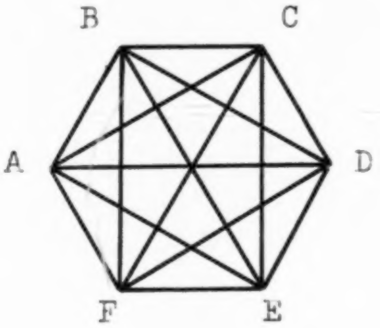
   (d) In 3-space, two lines each parallel to a given line through a point not on that line.

   Notice that "draw" is used to mean sketch or create without the exact instruments of straight-edge and compasses. The latter job is always referred to as "construct".

   The exercises above can play a valuable role in providing a place to discuss the place of "model" in geometry. We are trying to lead the student to accept the parallel property (or postulate) later in his life. Can we do this by implying that this geometry we are now developing is *one* model for the universe they live in, for many purposes the *best* model, but not the *only* model. Probably other, non-Euclidean models should not be mentioned at all, but just some leaving open of the door for the study of such geometries later without, at that time, creating a feeling that we have destroyed all the geometry previously taught and accepted.

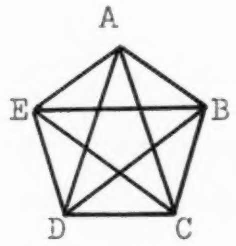
2. Work with regular polygons similar to this.

   (a) Consider the following regular hexagon and its diagonals:



Is there a side parallel to $\overline{AB}$?
Is there a diagonal parallel to $\overline{AB}$?
Is there a side parallel to $\overline{AD}$?
Is there a diagonal parallel to $\overline{AD}$?
Is there a side parallel to $\overline{CE}$?
Is there a diagonal parallel to $\overline{CE}$?

   (b) Consider the following regular pentagon:

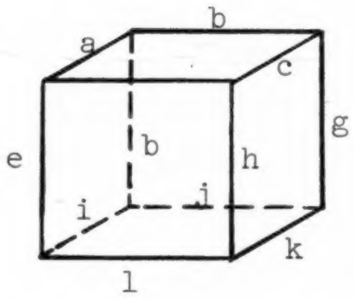


Is there a diagonal parallel to $\overline{AB}$?
Is there a side parallel to $\overline{AB}$?
Is there a diagonal parallel to $\overline{AD}$?
Is there a side parallel to $\overline{AD}$?

   (c) For which regular polygons will there be a side parallel to a side? a side parallel to a diagonal? a diagonal parallel to a diagonal?

3. Draw a diagram of a cube and letter the edges from $\underline{a}$ to $\underline{\ell}$. Start by considering a and b, then a and c, then a and d, etc., and complete this table:

x intersects y	x is parallel to y	x is skew to y
a intersects b	a is parallel to d	a is skew to g
a intersects d		
a intersects e	a is parallel to i	a is skew to h
a intersects f		
b intersects c		
etc.	etc.	etc.



Note: There are 66 statements to classify.

4. Two segments "at random".

A boy starts to invent a game in which he holds a straw in each hand and drops them simultaneously onto a table top from a height of at least one foot. If the straws cross each other he wins a point. Two boys then play in turn but find that the game leads to some difficulties which need to be resolved:

(1) Does "cross" mean "intersect" or must a point of one straw be between the endpoints of the other straw?

(2) Did you really mean the segments were intersecting or that the lines containing them were? What happens to the game if you choose the latter meaning?

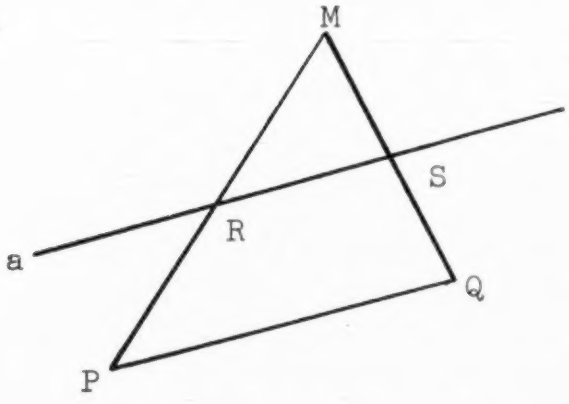
(3) What happens to the game if one straw may be thrown and then the other? What is the combination of skill and luck which is involved?

(4) Let us actually perform the game and make some tables to estimate probability. Would the game be the same if the two straws were different lengths? If there were two straws of each of the two different lengths? or 10 straws of each of these two lengths? or 10 straws all the same length? What do you mean by probability in these examples, anyhow?

(5) Discuss the relationship between the game and the mathematical model of the segments. It will soon become evident that the rules of the game evolve as one tries to play it, and that the geometric model by segments or lines is a good way to discuss what you want to agree upon for the rules. If one felt like it, one could mention that this is also the way that geometry itself developed: the agreements that people made, the procedures which are used developed through centuries as people did geometry. This might avoid the impression that someone wrote geometry from page one on as a great work of art and we still study it. In this sense our game with straws could be a model for geometry.

However, it must be admitted that the other direction was the reason for inventing the game. Line segments "at random" in a plane is a mathematical model of the game we are trying to invent.

5.



Line a is parallel to $\overline{PQ}$ and intersects side $\overline{MP}$ of $\triangle MPQ$ at its midpoint. What does it do to side $\overline{MQ}$? Suppose $MR = \frac{1}{3} MP$; what can you say about MS?

Section 2. Parallel two-dimensional objects:

2.1 Plane parallel to a plane (imitate Section 1.1).

2.2 Discuss absence of "skew" planes.

2.3 Line parallel to a plane; defined as non-intersecting.

2.4 Extend concept so that any two objects in the following list are defined as parallel: line, segment, ray, plane, half-plane; all by means of carrying lines and planes.

2.5 Two parallel lines (rays, segments) determine a plane.

2.6 Equations of lines parallel to coordinate axes; inequalities for strips and 2-space intervals.

Typical Exercises:

1. A very useful type of exercise for this part of geometry is the always-sometimes-never exercise. The following example will illustrate a little of what can be done with it:

   It is assumed that geometric objects are given with the relations among them stated in the hypothesis of each problem below. With the hypothesis of the problem decide whether the fact given in the conclusion will always be true, sometimes be true or never be true. Encircle the letter to indicate your decision with the following meanings:

   A The conclusion is always true.
   S The conclusion is sometimes true.
   N The statement is never true.

   A S N (1) Hypothesis: Two planes are parallel.
   Conclusion: A line in one of these planes is parallel to the other.

   A S N (2) Hypothesis: Two lines are parallel.
   Conclusion: A plane containing one of these lines is parallel to the other.

A S N (3) Hypothesis: A line is parallel to a plane.
Conclusion: A plane containing the line is parallel to the plane.

A S N (4) Hypothesis: A line intersects a plane.
Conclusion: A plane containing the line is parallel to the plane.

A S N (5) Hypothesis: A line intersects a plane.
Conclusion: A plane containing the line intersects the plane.

A S N (6) Hypothesis: Two parallel lines are each parallel to a plane.
Conclusion: The plane containing the lines is parallel to the plane.

A S N (7) Hypothesis: Two intersecting lines are parallel to a plane.
Conclusion: The plane containing these lines is parallel to the plane.

A S N (8) Hypothesis: Two planes are each parallel to a line.
Conclusion: The planes are parallel to each other.

A S N (9) Hypothesis: Two lines are each parallel to a plane.
Conclusion: The lines are parallel to each other.

A S N (10) Hypothesis: A line is parallel to one of two parallel planes.
Conclusion: The line is parallel to the other plane.

2. Consider a circle in a plane and a line intersecting the plane. Now consider the set of all lines which contain a point of the circle and are also parallel to the fixed line. Finally, consider a second plane parallel to the first plane. What is the set of segments between these two planes? Draw a sketch of this situation. Try with triangle; with quadrilateral.

3. Use patterns of prisms, exercises to go from easy to hard, first examples to supply some patterns and later examples to ask for the students to make their own. Here are some typical ones to discuss with a class:

   (a) Supply pattern of box.

   (b) Supply pattern for doubly-oblique rectangular prism.

   (c) Ask for pattern for singly-oblique prism.

   (d) Give pattern of regular pentagonal prism.

   (e) Ask for pattern for regular hexagonal prism.

4. What is the graph of $x = 2$ in 1-space? in 2-space?

5. What is the graph of $1 \leq x \leq 3$ in 1-space? in 2-space?

6. What is the graph of $1 \leq x \leq 3$ and $5 \leq y \leq 10$ in 2-space?

7. What is the graph of $x > 0$ in 1-space? in 2-space? What is the graph of $x > 0$ and $y > 0$ in 2-space?

Section 3. Transversals:

3.1 Review: Defs of vertical angles, adjacent angles, linear pair, complementary angles, supplementary angles; and facts about them:

   (1) Vertical angles congruent.

   (2) Complements of congruent angles are congruent; supplements of congruent angles are congruent.

   (3) Linear pair is supplementary; converse not necessarily true.

3.2 Defs: transversal lines to two lines in 2-space, in 3-space; to two planes in 3-space.

3.3 Defs: transversal planes to lines and planes in 3-space.

3.4 Def: dihedral angles.

3.5 Defs: in 2-space and 3-space: corresponding angles, alternate interior angles.

3.6 Parallel property: Through point not on line, one and only one parallel line exists.

(There was some doubt in the group about the wisdom of studying the parallel property at this time. Students have already much intuition about this property, but need it be stated in the text. Others in the group thought it wise to spell out the fact and that it was time to do so. No one meant that we would introduce a tight deductive system beginning with this postulate at this time. That is one reason we use the word "property" rather than "postulate".)

3.7 Properties in 2-space and in 3-space:

(1) parallel → congruent angles

(2) congruent angles → parallel

These apply to both corresponding and to alternate interior angles.

3.8 Construction: Line parallel to line through fixed point.

3.9 Defs: parallelogram, rhombus, trapezoid[*]; not extended to 3-space; also rectangle and square not used formally, but saved for later discussion of perpendicularity in Grade 8.

"Probably defined with only one pair of sides parallel".

3.10 Prove some theorems about quadrilaterals as example of deductive sequence. See below for suggestions.

3.11 Ask for some constructions and discuss sufficient data.

3.12 Segment parallel to side of a triangle; ratio of segments, ratio of areas.

Typical Exercises:

1. More ASN problems, for example: (Remind we are always in 3-space.)

   A S N (1) Hypothesis: A line intersects one of two parallel lines.
   Conclusion: The line intersects the other parallel line.

   A S N (2) Hypothesis: A line intersects one of two parallel planes.
   Conclusion: The line intersects the other parallel plane.

   A S N (3) Hypothesis: A plane intersects one of two parallel planes.
   Conclusion: This plane intersects the other parallel plane.

   A S N (4) Hypothesis: A plane intersects one of two parallel lines.
   Conclusion: The plane intersects the other parallel line.

2. Ask students to invent definitions for some concepts before teacher or book presents them: dihedral angle, vertical dihedral angles, adjacent dihedral angles, linear pair of dihedral angles, complementary dihedral angles, supplementary dihedral angles, betweenness for half planes.

3. Suggestions for theorems to be proved in deductive sequence:

   (a) Diagonal of parallelogram creates two congruent triangles.

   (b) Opposite sides of parallelogram are congruent.

   (c) If opposite sides of quadrilateral are congruent, then figure is parallelogram.

(d) Diagonals of parallelogram bisect each other.

(e) If diagonals of quadrilateral bisect each other, then figure is a parallelogram.

This about all. These will serve to review and keep alive the ideas of congruent triangles. Also, save for Grade 8 these:

(f) Diagonals of rhombus are perpendicular.

(g) Proof for sum of angles of triangle is 180.

4. Require constructions from given data and careful paragraph describing what has been done. Note the need of distinguishing and discussing problems which have insufficient data and thus lead to more than one solution, problems which have contradictory data, and problems which have a unique solution. Here are some typical examples:

   (a) Rhombus, given a side and an angle.

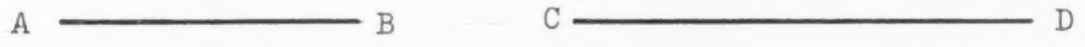
   (b) Parallelogram, given two sides (which?).

   (c) Rhombus, given two diagonals and one side.

   (d) Trapezoid, given one side, one diagonal, and one angle.

A few problems to show the direction of development and the slope of the incline of difficulty might illustrate what is meant here. Let us expand the meaning of item (b) on the preceding list.

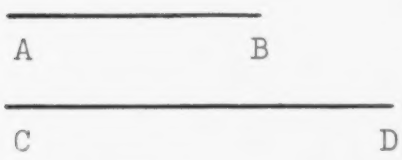
(1) Can you have a parallelogram with two adjacent sides congruent to these segments:

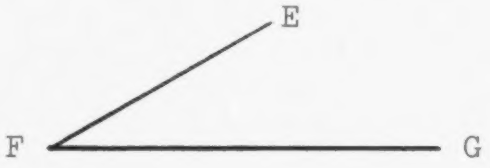


If so, construct such a parallelogram.

(2) Can you have more than one parallelogram with the data given in problem (1)? If so, construct a parallelogram of this sort which is not congruent to that which you constructed for (1).

(3) Can you have a parallelogram with two adjacent sides congruent to the segments below, and the angle between these sides congruent to the angle below:



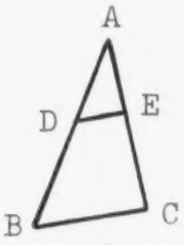


If so, construct such a parallelogram.

(4) Can you have more than one parallelogram with the data given in problem (3)? If so, construct a parallelogram of this sort which is not congruent to the one you constructed for (3).

5. Construct a figure similar to a given figure starting with the following: rhombus, parallelogram, trapezoid.

6.

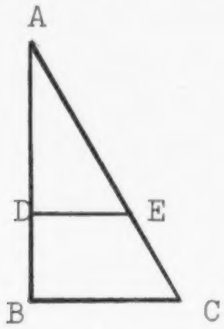


$\overline{DE} \parallel \overline{BC}$, Ad = 3, DE = 2

DB = 4, Find BC

(Easier ones at first, of course.)

7.



$\angle ADE$ and $\angle ABC$ are right angles, so we also have $\overline{DE} \parallel \overline{BC}$; AD = 8, DE = 6, AB = 12.

(a) Find the area of $\triangle ADE$ and area of $\triangle ABC$.

(b) Find area of trapezoid DEBC.

(c) What is the ratio of the areas of $\triangle ADE$ and $\triangle ABC$?

Section 4. Transversals to three or more lines and planes:

4.1 Three or more coplanar parallel lines and transversal lines.

4.2 Three parallel lines and transversal planes.

4.3 Three parallel planes and transversal lines; also transversal planes.

4.4 Intuitive understanding of segments cut off by and on such transversal lines.

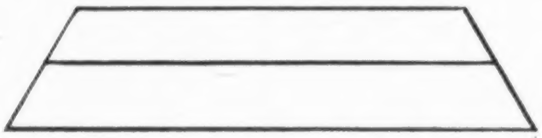
4.5 Median of trapezoid and relationship to diagonals.

4.6 Show connection between nets of parallel lines and parallel planes to coordinate systems in 2 and 3 dimensions; do not restrict to perpendicular sets of lines and planes.

Typical Exercises:

1. Show how to divide segment into 7 (or any number of parts) congruent segments by edge of ruled sheet of paper.

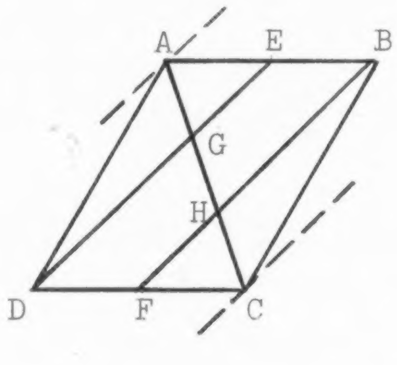
2. Median of trapezoid bisects each diagonal.



3. E is midpoint; F is midpoint.

   Prove: AG = GH = HC

   Should we give hits of dotted lines?
   Develop this problem; do not just throw it at them.



CONTENTS OF GRADE 8

Sequence A

Chapter 1: Perpendicularity

Section 1: Perpendicularity of One-dimensional Objects

- 1.1 Right angle and perpendicular lines
- 1.2 Extension to line, ray, segment by concepts of "carrying line"
- 1.3 Construction of perpendicular line through fixed point
- 1.4 Construction of bisector of segment
- 1.5 Locus in plane of points equidistant from two fixed points
- 1.6 Proof concerning diagonals of rhombus

Section 2: Perpendicularity of Two-dimensional Objects

- 2.1 Dihedral angles
- 2.2 Extension of perpendicularity from planes to half-planes by concept of "carrying plane"
- 2.3 Locus in 3-space of points equidistant from two fixed points
- 2.4 Definition of line perpendicular to plane
- 2.5 Perpendicular skew lines
- 2.6 Line perpendicular to two lines at some point: determination of plane
- 2.7 Mutually perpendicular lines; same for planes
- 2.8 Plane through fixed point, perpendicular to fixed line

Chapter 2: Coordinate Systems - Distance

Section 1. One Dimensional Coordinate System

- 1.1 Coordinate on a line
- 1.2 Distance between two points on a line
- 1.3 Algebraic description of subsets of the line

Section 2. Two Dimensional Coordinate System

- 2.1 Coordinates in the plane
- 2.2 Distance between two points in the plane

2.3 Algebraic description of subsets of the plane
Section 3. Three Dimensional Coordinate System
3.1 Coordinates in Space
3.2 Distance between two points in space
3.3 Algebraic description of subsets of space
Section 4. Polar Coordinate System

Chapter 3: Displacements
Section 1. Physical Quantities
1.1 Quantities
1.2 Operations (review)
Section 2. Vector Quantites
Section 3. Vectors
3.1 Activities
3.2 Equality
3.3 Opposite of a vector
3.4 Addition of vectors
3.5 Zero vector
3.6 Commutative principle for vector addition
3.7 Associative principle for vector addition
3.8 Summary of properties of addition
3.9 Solution of vector equations
Section 4. Multiplication of a Vector by a Number
4.1 Developing meaning of multiplication
4.2 Multiplication and parallelism
Section 5: Translation
Section 6: Decomposition
6.1 Decomposing in terms of two vectors
6.2 Naming vectors
Section 7: Extension to Vectors in 3-Space
Section 8: Applications

Chapter 4: Problem Analysis (Strategies)

- Section 1: Translation of Phrases
  - 1.1 Mathematical phrases to English phrases
  - 1.2 Class discussion
  - 1.3 Characteristics of translations of English phrases
  - 1.4 Exercises
  - 1.5 English phrases to mathematical phrases
  - 1.6 Class discussion
  - 1.7 Characteristics of translations of mathematical phrases
  - 1.8 Exercises
- Section 2: Translation of Sentences
  - 2.1 Mathematical sentences to English sentences
  - 2.2 Class discussion
  - 2.3 The translation process
  - 2.4 Characteristics of translations to English sentences
  - 2.5 Sentences involving restricted domain
  - 2.6 Exercises
  - 2.7 English sentences to mathematical sentences
  - 2.8 Class discussion
  - 2.9 Exercises
- Section 3: Problem Analysis and Strategies
  - 3.1 Basic attitudes toward problem analysis
  - 3.2 Organization techniques (a first strategy)
  - 3.3 Example of first strategy at work
  - 3.4 A second strategy
  - 3.5 Exercises
  - 3.6 Organizing information with drawings or diagrams
  - 3.7 Exercises
  - 3.8 Organizing information in tabular form
  - 3.9 Exercises
  - 3.10 Estimation process
  - 3.11 Exercises
  - 3.12 Problem analysis based on analogy

Chapter 5: Number Theory

Section 1: Even and Odd Integers
2: Informal Discussion of Statements and Proof
3: Factors, Divisibility, Tests for Divisibility, and the Division Algorithm
4: Prime Numbers, Sieve of Eratosthenes, Prime Factorization
5: The Euclidean Algorithm and the GCD

Chapter 6: The Real Numbers Revisited - Radicals

Section 1: Motivation
Section 2: Review of Facts about the Real Number system
- 2.1 Notation for real numbers

Section 3: Roots of Numbers
- 3.1 Square roots
- 3.2 Definition of the n-th root of a
- 3.3 (possibly) Introduce $x^{1/2}$, $x^{1/4}$, etc.

Section 4: Computation with Radicals
- 4.1 Use of factorization to find roots
- 4.2 Irrational square roots
- 4.3 Product of square roots
- 4.4 Square roots of rational numbers

Section 5: Review of Real Number Properties and the Number Line
- 5.1 Properties of the real number system
- 5.2 Real numbers and the number line

Chapter 7: Truth Sets of Mathematical Sentences

Section 1: Addition and Multiplication Properties of Equality and Inequality
- 1.1 Concept of equivalent sentences
- 1.2 Addition property of equality
- 1.3 Multiplication property of equality
- 1.4 Addition and multiplication properties of inequalities
- 1.5 Applications to verbal problems

Section 2: Permissible Operations for Equivalent Sentences

- 2.1 Addition and multiplication
- 2.2 If $a = b$, then $a^2 = b^2$; converse not true
- 2.3 Use of "$ab = 0$ if and only if $a = 0$ or $b = 0$"
- 2.4 Restrictions on denominators containing veriables
- 2.5 Squaring both sides of an equation

Chapter 8: Quadratic Polynomials as Functions

Section 1: Graph of the Quadratic Function

- 1.1 Graphs of $f : x \rightarrow x^2$ and $f : x \rightarrow -x^2$
- 1.2 Graph of $f : x \rightarrow ax^2$
- 1.3 Graph of $f : x \rightarrow ax^2 + k$
- 1.4 Graph of $f : x \rightarrow a(x - h)^2$
- 1.5 Graph of $f : x \rightarrow a(x - h)^2 + k$
- 1.6 Point out need to factor quadratic polynomials

Section 2: Factoring Polynomials

- 2.1 Meaning of "over the integers", etc.
- 2.2 Type $ab + ac = a(b + c)$
- 2.3 Type $ax + ay + bx + by = (a + b)(x + y)$
- 2.4 Perfect squares
- 2.5 Difference of squares
- 2.6 Type $x^2 + bx + c$, by completing the square
- 2.7 Type $ax^2 + bx + c$, by completing the square, and by inspection

Section 3: Solving Quadratic Equations

- 3.1 Same as "finding zero of function"
- 3.2 Solution by factoring, including completing the square
- 3.3 Formula as a short cut

Section 4: Writing the General Quadratic in Form $a(x - h)^2 + k$

- 4.1 Completing the square to get the form
- 4.2 Use the graph of a single quadratic function to solve many related quadratic equations

Chapter 9: Probability

Section 1: Dependent and Independent Events
2: Conditional Probability - Bayes' Theorem (WST)
3: Expectation
4: Variation, Standard Deviation
5: Normal Distritution; Physical Observations

Chapter 10: Parallels and Perpendiculars

Section 1: Regions
1.1 Seperation of a plane by parallel lines
1.2 Separation of a plane by n parallel lines and m others perpendicular to them
1.3 Extension to 3-space with parallel and perpendicular planes
Section 2: Combining Parallel and Perpendicular Relations
2.1 Line perpendicular to one of two parallel lines (planes)
2.2 Two lines perpendicular to the same line (plane)
2.3 Plane perpendicular to one of two parallel lines (planes)
2.4 Two planes perpendicular to same line (plane)
2.5 Relation of parallel and perpendicular with respect to reflexive, symmetric, transitive relations
Section 3: Distance between Parallel Lines and Parallel Planes
3.1 (Review) Distance between two points
3.2 Distance from a point to a set of points
3.3 Distance between two sets of points
3.4 Altitude of parallelogram, of trapezoid
3.5 Equations and inequalities for planes parallel to co-ordinate planes.
Section 4: The Quadrilateral Properties
4.1 Review properties of sides, diagonals, angles of figures
4.2 Informal approach to "necessary and sufficient"

- Section 5: Symmetries
  - 5.1 Symmetry in a line (in 2-space and in 3-space)
  - 5.2 Symmetry in a point (in 2-space and in 3-space)
  - 5.3 Symmetry in a plane (in 3-space)
  - 5.4 Symmetries of triangles
  - 5.5 Symmetries of rectangles
  - 5.6 Symmetries of a circle
  - 5.7 Symmetries of 3-dimensional figures
- Section 6: Angle-Sum Proofs
  - 6.1 The parallel property
  - 6.2 Angle measure sum for triangles
  - 6.3 Angle measure sum for convex polygons

Chapter 11: Properties and Mensuration of Geometric Figures

- Section 1: Motivation of Numerical Measure for Areas
- Section 2: Arbitrary Unit versus Standard Unit
  - 2.1 Selection of unit
  - 2.2 Metric system
- Section 3: Assigning Measures to Segments and to Regions
  - 3.1 Formulas for perimeters and areas
  - 3.2 Measure and Congruence
- Section 4: Properties of Regular Polygons
- Section 5: Models of Solids
- Section 6: The Sphere
  - 6.1 Surface of the sphere
  - 6.2 Volume of the sphere

Chapter 12: Spatial Perception and Locus

- Section 1: Relationships between two (or more) Given Point Sets
  - 2: Using a Set of Points to Evolve Another Set of Points
  - 3: Sets of Points Meeting Given Conditions

Chapter 13: Systems of Equations in Two Variables

Section 1: Solution Sets of Systems of Equations and Inequalities
    1.1 Review definition of solution set of equation or inequality
    1.2 Define solution set for system of equations or inequalities
Section 2: Equivalent Equations; Equivalent Systems
    2.1 Definition of equivalent as "having same solution set"
    2.2 Replacement of an equation in a system by an equivalent equation results in an equivalent system
    2.3 Linear combination method of solution
Section 3: Systems of Linear Equations
    3.1 Review of graphical solution
    3.2 Graphical interpretation of linear combination method
Section 4: Graphical Solution of Systems of Inequalities
Section 5: Applications
    5.1 Word problems needing two variables
    5.2 Use of mathematical models
    5.3 Introduction to linear programming

GRADE 8 - CHAPTER 1

PERPENDICULARITY

Background:

Many of the ideas of perpendicularity of lines are already known to the student at this time, but the careful extension of this concept to perpendicular rays and segments is probably new. The concept of dihedral angle has been used and discussed, but it is here extended to define perpendicular planes. The further extension to perpendicular half planes and of line perpendicular to plane are quite new but are illustrated by familiar facts about cubes.

Simple constructions have been introduced to copy segments and angles and to bisect an angle, but here the constructions of line perpendicular to line and point bisecting segment are introduced and proved.

It is hoped that the idea of locus is not new, but its application to perpendicular bisecting line and perpendicular bisecting plane of a segment is formalized.

The rhombus and square are known so their perpendicular diagonals should not need much time to introduce.

Purpose:

The purposes of this chapter are as follows:

1. To strengthen, and sometimes to define, the concept of perpendicular relations in 2 and 3 dimensions.
2. To illustrate the concept of perpendicularity for 1 and 2 dimensional figures.
3. To associate 2 and 3 dimensional concepts as often and as fully as possible at this level of mathematics.

4. To introduce some new constructions with straight edge and compasses and to review old constructions in harder problems.
5. To illustrate the idea of locus in perpendicular bisecting lines and planes without trying to teach the general idea of locus.
6. To provide much inductive work with perpendicularity in 3 dimensions, but no proof.
7. To introduce 3 mutually perpendicular lines and 3 mutually perpendicular planes as preparation for coordinate geometry in 3-space.

Rationale:

Parallelism has been introduced by itself in a former chapter. Now another basic relationship among lines and planes in 2 and 3-space is isolated from other relationships. It is both a summary and an extension of ideas about perpendicularity.

The discussion of the interrelationships between the concepts of parallelism and perpendicularity are summarized in a later chapter. Thus we have just one more example of the planned spiral approach in the curriculum. Many of the properties of the triangle and quadrilateral are being introduced all along the way to prepare for summaries of such facts later. It is clear that some ideas are introduced in the present chapter to prepare for coordinates in 3-space in the chapter which follows immediately. Some future topics which will use the ideas of perpendicularity are these: mensuration of geometric figures, transformations, and tangency.

Section 1. Perpendicularity of one-dimensional objects:

1.1 Right angle as degree measure 90; perpendicular lines as containing a right angle.

1.2 Extend perpendicular to other one-dimensional objects (line, ray, segment in all combinations) by concept of carrying line; note that perpendicular does not imply intersect.

1.3 Construction: line through fixed point perpendicular to fixed line (whether point is on line or not); proof by congruent triangles.

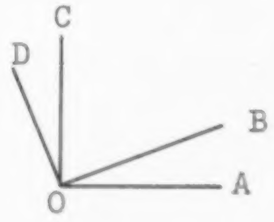
1.4 Construction: bisector of segment; proof by congruent triangles.

1.5 Locus in a plane; points equidistant from two fixed points.

1.6 Prove: Each diagonal of a rhombus is the perpendicular bisector of the other diagonal; therefore true for square; but not true for non-square rectangle.

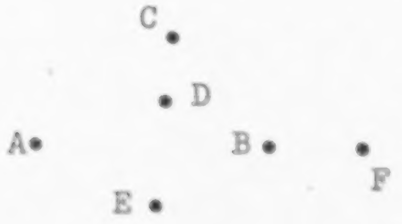
Typical Exercises:

1.



$\angle AOC$ is a right angle; $\angle BOD$ is a right angle; $m\angle AOB = 30$. What is $m\angle COD$?

2.



Given that $\overleftrightarrow{AB} \perp \overleftrightarrow{CD}$: A, B and F are collinear; C, D and E are collinear; and order and separation as indicated by the figure.

Answer these questions: Encircle either T or F to indicate that the fact is true or false:

T F (1) $\overleftrightarrow{AB}$ intersects $\overleftrightarrow{CE}$
T F (2) $\overleftrightarrow{AB} \perp \overline{CD}$
T F (3) $\overleftrightarrow{AB}$ intersects $\overline{CD}$
T F (4) $\overleftrightarrow{AB}$ intersects $\overline{CE}$
T F (5) $\overline{AB}$ intersects $\overline{CD}$
T F (6) $\overline{AB} \perp \overline{CD}$

T F (7) $\overrightarrow{AB}$ intersects $\overrightarrow{CD}$

T F (8) $\overrightarrow{BF} \perp \overrightarrow{DC}$ etc.

3. More ASN questions similar to these:

   (a) Hypothesis: Object $\emptyset \perp$ Object ψ
   Conclusion: Object $\emptyset$ intersects Object ψ

   (b) Hypothesis: Object $\emptyset$ intersects Object ψ
   Conclusion: Object $\emptyset \perp$ Object ψ

   (c) Hypothesis: $\angle ABC$ is a right angle.
   Conclusion: $\overrightarrow{AB} \perp \overrightarrow{AC}$

   (d) Hypothesis: $\angle ABC$ is a right angle.
   Conclusion: $\overrightarrow{BA} \perp \overrightarrow{BC}$

   (e) Hypothesis: $\overrightarrow{BA} \perp \overrightarrow{BC}$
   Conclusion: $\angle ABC$ is a right angle.

4. Use the following exercise to prepare for Section 1.3 above: If $PA = PB$ and $QA = QB$, then $\overleftrightarrow{PQ} \perp \overleftrightarrow{AB}$. Assume all points mentioned are collinear.

5. The same hypothesis given in Exercise 4 could prepare for Section 1.4 above by proving that $\overleftrightarrow{PQ}$ bisects $\overline{AB}$.

6. Prove this theorem in connection with Section 1.5: If $PA = PB$, $QA = QB$ and R is on $\overleftrightarrow{PQ}$, then $RA = RB$. Note that this fact is also true in 3-space.

7. Work up to such construction problems as these as objectives:

   (a) Square, given a side.

   (b) Rectangle, given two sides.

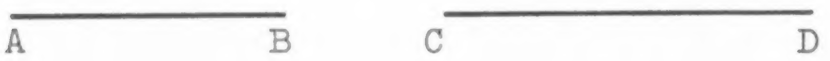
   (c) Rhombus, given two diagonals.

   (d) Parallelogram, given two diagonals.

   (e) Rectangle, given a diagonal and a side.

To show how these types of problems may suggest thinking about sufficiency of data, consider these examples:

(1) Can you have a quadrilateral with two diagonals congruent to these segments?



If so, construct such a quadrilateral. Can you have more than one such quadrilateral? If so, construct another which is not congruent to the first.

(2) Can you have a parallelogram with diagonals congruent to the segments in Exercise (1)? Construct such a parallelogram. If you can have another, non-congruent parallelogram, construct one.

(3) Can you have a quadrilateral with diagonals perpendicular to each other and congruent to the segments in Exercise (1)? If so, construct one. Can you have another, non-congruent quadrilateral which fits these conditions? If so, construct one.

(4) Can you have a rhombus with diagonals congruent to the segments in Exercise (1)? If so, construct one. Can you have another, non-congruent rhombus which fits these conditions? If so, construct one.

(5) Can you have a square with diagonals congruent to the segments in Exercise (1)? If so, construct one. Can you have another, non-congruent square which has such diagonals? If so, construct one.

8. Extend the ideas of paper folding used in previous chapters. Here use this technique to find the altitudes and perpendicular bisectors of sides of triangles. Use a helpful selection of triangles from scalene, isosceles and equilateral; acute, right and obtuse.

9. Start with four different triangles and use them to do the following constructions (with ruler and straight edge):

   (a) Construct the three altitudes.

   (b) Construct the three medians.

   (c) Construct the three angle bisectors.

   (d) Construct the three perpendicular bisectors of the sides.

   Use different size and different shape triangles within the class or even for each pupil. Be sure to have some obtuse triangles to construct altitudes and perpendicular bisectors of sides.

Section 2. Perpendicularity of two-dimensional objects:

2.1 Defs: measure of dihedral angle, right dihedral angle, perpendicular planes as containing a right dihedral angle.

2.2 Extend perpendicular to two dimensional objects (planes or half-planes) by concept of carrying plane; note that perpendicular does not imply intersects.

2.3 Locus in 3-space; points equidistant from two fixed points.

2.4 Def: line perpendicular to plane as perpendicular to all lines in plane through its foot; extend to one-dimensional objects (line, segment, ray) perpendicular to two-dimensional objects (plane, half-plane); perpendicular does not imply intersects.

2.5 Discuss perpendicular skew lines, perhaps by means of line to which each of them is perpendicular; extend to line perpendicular to two lines and intersecting them at distinct points.

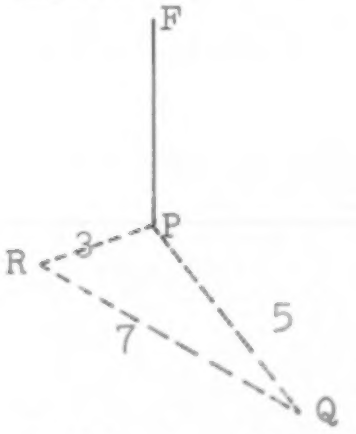
2.6 Line perpendicular to two lines and intersecting them at the same point; line perpendicular to plane determined by two intersecting lines (no proof of any of this at this stage).

2.7 Three mutually perpendicular lines; three mutually perpendicular planes.

2.8 Plane containing fixed point and perpendicular to fixed line (whether line contains point or not).

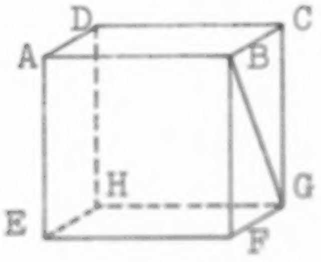
Typical Exercises:

1.



$\overline{FP}$ is a flagpole standing on level ground at point P, R and Q are points on the ground so that RP = 3, PQ = 5, and RQ = 7. Copy the figure and add the line which is perpendicular to both $\overleftrightarrow{FP}$ and $\overleftrightarrow{RQ}$.

2. The following figure is a cube. By means of the figure answer the questions.



(a) $\overline{AE}$ is skew to $\overline{DC}$. Name a segment perpendicular to both and intersecting both.

(b) $\overline{AE}$ is skew to $\overline{DC}$. Name a segment perpendicular to both, but not intersecting either.

(c) Find a segment skew to $\overline{BG}$ such that there is a segment marked in this figure which is perpendicular to both. Name the segment perpendicular to both.

(d) Find a line perpendicular to two intersecting lines so that the intersecting lines are not perpendicular to each other.

(e) Describe the plane through C perpendicular to $\overleftrightarrow{CG}$.

(f) Describe the plane through E perpendicular to $\overleftrightarrow{HG}$.

(g) Describe the line through D perpendicular to the plane DCGH.

(h) Describe the line through F perpendicular to the plane ADHE.

3. Draw a picture of a cube and label the eight vertices.

   (a) Using these letters name three lines such that each pair is perpendicular.

   (b) Identify three planes in the figure such that each pair is perpendicular.

   (c) Identify a line perpendicular to two intersecting lines.

   (d) Identify a line perpendicular to a plane.

GRADE 8 - CHAPTER 2

COORDINATE SYSTEMS - DISTANCE

Background Assumed:

Knowledge of the real number line.

Familiarity with coordinates of points on a line and coordinates of points in a plane and how to plot them.

Ability to graph linear functions and $y = |x|$.

From Grade 7, Chapter 5, measure of segments.

From Grade 7, Chapter 11, some introductory knowledge of:

(1) equations of lines parallel to the coordinate axes, inequalities for stripe, and 2-space intervals.

(2) Set-builder notation, e.g., $\{(x,y) : x + y > 2\}$.

Purpose:

To develop mathematical machinery:

(1) to describe algebraically sets of points that form familiar geometric figures, and, conversely, to describe geometrically solution sets of equations,

(2) to be able to prove geometrical theorems analytically,

(3) to solve graphically many problems that are usually solved algebraically.

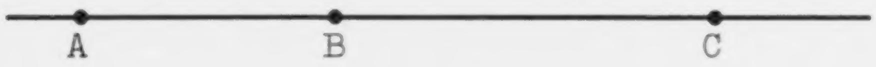
This chapter brings together algebra and geometry and lays the foundation for the study of a body of mathematics called analytical geometry.

Section 1. One dimensional coordinate system:

1.1 Distance

Intuitive development of the concept of distance. What is the distance between San Francisco and Los Angeles? This is a tough question, because we wonder which road, by airplane exactly where in San Francisco do we start and exactly where do we stop and what is the unit of distance. How far from Joe's house to Bill's house? It should be clear from such a short discussion that we will have difficulty getting what everybody would agree to be an "exact" answer.

Consider three collinear points:



Have students measure the length of $\overline{AB}$, $\overline{BC}$, and $\overline{AC}$ using some unit, and have them notice the apparent relationship between the three lengths. Connect this up with the relation betweenness. The number associated with $\overline{AB}$ is called the distance between A and B with respect to that unit and is denoted by AB.

Consider three non-collinear points and make measurements as above to motivate the triangle inequality, $AB + BC > AC$.

From the above it should be possible to pull out the idea that given two points A and B in space, and a unit of measure, then there is a number that corresponds to these two points. This number is the distance between A and B and is denoted by AB.

This correspondence, then, is a function, D,

$$D : \text{(points) } A,\ B \rightarrow \text{(distance)} d\ .$$

Consider what happens to the distance between A and B as point B gets closer and closer to point A.

This leads us to 0 as the distance between point A and point A, that is $\overline{AA} = 0$.

Note here that the distance between points A and B is either positive or zero; it is zero whenever A and B are names for the same point, and positive whenever $A \neq B$.

1.2 Coordinates on a line. (See Mathematics for Junior High School, Vol. 2, Part 1, pp. 21-23.)

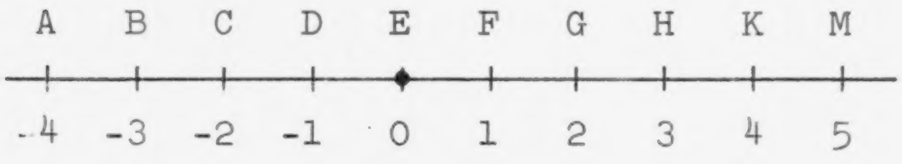
One to one correspondence between the points on a line and the real numbers giving the real number line.

Show how to coordinatize the line by choosing an arbitrary point as the origin, and an arbitrary unit.

1.3 Distance between two points on a line.

The coordinate of a point on a line gives the distance from the origin to the point and also tells the direction from the origin to the point.

The absolute value of a real number, a, gives the distance between the origin and point A with coordinate a. Consider the number line:



Find the distances FK, AD, and CH.

Notice that if P and Q have coordinates p and q respectively that $PQ = |p - q|$.

At this point it might be well to consider two distinct points P and Q with coordinates p and q and examining the numbers $p - q$ and $q - p$. One of these is positive and one is negative. This can be connected up with the concept of sense on line and also with the concept of directed distance. $q - p$ is the directed distance from P to Q and it is positive if the positive sense of the line is in the direction from P to Q.

If this informal bit on directed distance does not get into the text, it should be noted in the TC for the teachers.

Define: midpoint of a segment.

Find the coordinate of the midpoint of segments $\overline{FM}$, $\overline{AE}$, $\overline{BF}$.

Develop formula for coordinate, m_0, of the midpoint of $\overline{PQ}$ if P, and Q have coordinates p and q. $m_0 = \frac{p+q}{2}$.

Exercises of the following type:

1. Given $P(2,5)$ and $Q(-4,3)$
   - (a) Find the coordinates of the midpoint of $\overline{PQ}$.
   - (b) Find the coordinates of the point R on $\overline{PQ}$ such that $PR = \frac{1}{4} PQ$.
   - (c) Find the coordinates of the point S such that Q is the midpoint of $\overline{PS}$.

1.4 Algebraic description of subsets of the line. (See Math. for Junior High School, Vol. 2, Part 1, pp. 65-71)

At this stage of the game we make the convention that if P has coordinate p in a given coordinate system, we may refer to P by simply naming its coordinate p.

Segment $\overline{FK}$ (in diagram, page 3) consists of the points F, K, and all points between F and K. Analytically we may say $\overline{FK} = \{x : 1 \leq x \leq 4\}$ or $\overline{FK}$ is the solution set of the sentence $1 \leq x \leq 4$.

Develop the following:

If $p < q$ segment : $\{x : p \leq x \leq q\}$
half line : $\{x : x > q\}$, $\{x : x < q\}$
ray : $\{x : x \geq q\}$, $\{x : x \leq q\}$

If $p < q$ open interval : $\{x : p < x < p\}$

Find the solution sets of the following sentences and graph the solution set -- name the set if it is a familiar geometric figure:

1. $x = 2$
2. $x - 14 = -12$
3. $x > 3$
4. $x + 7 \geq 10$
5. $x < -2$
6. $x - 4 \leq -6$
7. $3 < x < 7$
8. $-2 \leq x - 5 \leq 2$

9. x greater than 3 and x less than 7
10. x greater than 3 or x less than 7
11. x greater than 3 or x less than -2
12. $x \geq 3$ or $x \leq -2$

Describe the following sets of points, graph them and then name the set if its graph is a familiar figure.

13. $\{x : 5 < x < -2\}$
14. $\{x : x^2 < 4\}$
15. $x^2 \geq 4$
16. $|x - 2| = 3$
17. $|x + 7| = 5$
18. $|x| - 3 = 4$
19. $|x| + 7 = 2$
20. $|x - 2| < 4$
21. $|x| + 2 > 5$

etc.

Section 2. Two dimensional coordinate system:

2.1 Coordinates in the plane. (See Vol. 2, Part 1, pp. 23-30)

Set up the X-axis, Y-axis and review the plotting of points and discuss and note the one-to-one correspondence between points in the plane and the set of all ordered pairs of real numbers (x,y). This would be a good place to consider developing the concept of cartesian product $A \times B$ where A and B are sets of numbers.

Begin with $A = \{1,2,3\}$ and $B = \{4,5\}$ and form $A \times B$ and finally consider $R \times R$ where R is the set of all real numbers. Then, point out that for each $(x,y) \in R \times R$ there is a point in the plane and for each point P in the plane there corresponds an ordered pair of real numbers $(z,b) \in R \times R$.

Definition of the four quadrants showing how the two coordinate axes separates the plane into 4 disjoint sets of points.

Exercises of the following type.

1. Consider a square with side 5. If the x-axis and y-axis are sides of the square, what are the coordinates of the vertices of the square if one vertex is in the third quadrant? Consider the other quadrants.
2. Consider an isosceles triangle with base 6 and altitude 4, if the vertex is at the origin and the y-axis bisects the base what are the coordinates of the vertices of the triangle? Consider all cases.
3. Give an algebraic description of the set of points in each quadrant.
4. Consider two sentences of the form $y = 2x + 4$, graph these two sentences, these two lines separate the first quadrant into three convex regions. Find algebraic descriptions of each of these three regions.

2.2 Distance between two points in a plane.

Use ideas developed in 1.2 to find the distance between two points on the X-axis, Y-axis and on lines parallel to the axes.

Give problems finding the distances between points A and B that are in different quadrants.

Find the distance AB for:

1. A(0,7), B(0,-3)
2. A(-8,0), B(2,0)
3. A(8,5), B(8,-2)
4. A(-2,-3), B(-2,5)

Have students see that the formula developed in 1.2 can be used here.

As a discovery exercise, consider the problem: find AB for A(2,4) and B(6,1).

Review subscript notation such as $P_1(x_1,y_1)$, $P_2(x_2,y_2)$.

Develop proof of distance formula

$$P_1P_2 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \ .$$

Give problems so that the square roots involved are within the square root development in Grade 7. Keep the distances mostly rational, but have few that are irrational.

Some problems could involve finding areas of figures that are plotted in the plane.

Develop the formula for finding the coordinates of the midpoint of the segment joining $P_1(x_1,y_1)$ and $P_2(x_2,y_2)$. This can be done using similar triangles or using the distance formula just developed. It could be that the algebra involved here it too heavy for them to use the distance formula. The similar triangle method has simple algebraic manipulation.

Give problems using midpoint formula and distance formula.

Prove some originals and theorems from synthetic geometry by using coordinate geometry.

Exercises:

1. Consider $\triangle ABC$ with vertices $A(2,3)$, $B(5,9)$, $C(11,6)$.

   (a) Find the lengths of the sides AB, BC, and AC.

   (b) What kind of a triangle is $\triangle ABC$?

   (c) Find the coordinates of the midpoints of $\triangle ABC$.

   (d) Prove that the line joining the midpoints of segments $\overline{AB}$ and $\overline{BC}$ is parallel to $\overline{AC}$ and is half the length of $\overline{AC}$.

2. Consider $\triangle ABC$ with vertices $A(3,10)$, $B(3,4)$, and $C(9,4)$.

   (a) Find the lengths of the sides.

   (b) What kind of a triangle is it?

   (c) Find the midpoint D of AC and prove that $BD = \frac{1}{2} AC$.

3. Given $P(3,4)$. Find P^1 such that the origin O is the midpoint of PP^1.

4. Given $P(3,4)$ and $Q(6,8)$. Find 3 points equidistant from P and Q.

2.3 Algebraic description of subsets of the plane. (See Vol. 2, Part 1, pp. 30-33, 93 - 110)

1. Sentences in one variable.

   For k and m real numbers consider sentences of the types: $x = k$, $y = k$, $x > k$, $x \geq k$, $x < k$, $x \leq k$, $y > k$, $y \geq k$, $y < k$, $y \leq k$, $k < x < m$, $k \leq x \leq m$, $k \leq x < m$, etc.

   Judicious use of set notation sometimes clarifies things here $\{(x,y) : 2 < x < 5\}$.

   Consider the following sets of points; describe and name the geometric figure where possible.

   (1) $\{(x,y) : x > 2 \text{ and } y > 4\}$

(2) $\{(x,y) : x \geq 2 \text{ or } y > 4\}$

(3) half plane: $\{(x,y) : x > 4\}$

(4) strip: $\{(x,y) : 2 < x 5\}$

(5) rectangular region: $\{(x,y) : 2 \leq x \leq 4 \text{ and } -1 \leq y \leq 3\}$

Give problems involving absolute value and inequalities.

2. Sentences in two variables.

rectangular region: $\{(x,y) : 2 \leq x \leq 4 \text{ and } -1 \leq y \leq 3\}$

line: $\{(x,y) : 2 \times +y = 2\}$

Develop concept of slope of a line (see Vol. 2, pp. 374-378).

Use definition as rise over run and get slope formula for the line through $P_1(x_1,y_1)$ and $P_2(x_2,y_2)$, $m = \frac{y_2 - y_1}{x_2 - x_1}$.

Consider the lines $y = mx$ where m is a real number and discover that m is the slope.

Consider the type of line with $m > 0$, $m = 0$, $m < 0$, and slope undefined. Introduce terms horizontal and vertical lines.

Define x-intercept and y-intercept of a line.

Plot the line $y = 2x$ and then consider the line $y = 2x + 3$.

Discuss the family of lines $y = mx$ where m is a real number, $y = 2x + b$ where b is a real number.

Consider lines of the type $y = mx + b$.

Problem: Describe two lines which have no slope.

Develop the conditions for parallel lines by examining the character of the equations of lines which are parallel.

At this point, the student has the mathematical machinery to prove the theorem: Two non-vertical lines are parallel if and only if they have the same slope.

Consider the lines of the type $y = mx + b$ and $y = -\frac{1}{m}x + b$. Develop the conditions for perpendicular lines by examining the character of the equations of lines which are perpendicular.

At this point if the student can use the Pythagoream Theorem and its converse, he could prove that: Two lines neither of which is vertical are perpendicular if and only if the product of their slopes is -1.

Exercises:

This set of exercises should contain originals and proofs of theorems found in any standard analytic geometry text.

1. Consider the quadrilateral $A(0,0)$, $B(2,4)$, $C(7,4)$, and $D(8,0)$.
   (a) Prove that this quadrilateral is a trapezoid.
   (b) Join the midpoints of the sides and prove the figure is a parallelogram, and find the lengths of its sides.
2. Find p so that the line $px + 3y = 7$ is:
   (a) parallel to the line $y = 2x + 4$;
   (b) perpendicular to the line $y = -3x - 7$.
3. In the family of lines $y = mx + 4$, find the line parallel to $y = x$.
4. Describe geometrically and algebraically the set of points in the plane 3 units from the origin.
5. Describe geometrically and algebraically the set of points in the plane 3 units from the segment with endpoints $A(0,0)$ and $B(5,0)$.

Section 3. Three dimensional coordinate system:

3.1 Coordinates in space.

Use suitable ideas from Grade 8, Chapter 1, in introducing coordinates in space. Make liberal use of the math room as one octant and use the intersections of the floor and walls at one corner of the room to illustrate the three mutually perpendicular lines. Name the coordinate planes and the octants and show that the three mutually perpendicular planes determined by the three mutually perpendicular lines separate space into 8 disjoint regions. Note that the labelling of the coordinate axes is arbitrary. Many people prefer to label the axes in such a way that they form a right-handed coordinate system.

Develop one-to-one correspondence between the ordered triples (x,y,z) and the points in space.

3.2 Distance between two points in space.

1. Show that if the two points happen to both be on a coordinate axis or on one of the coordinate planes, then we already have developed a formula for finding the distance between the two points.

   Do some work in plotting points in three space.

   First, find the distance from the origin to a point in space, say to $P(\sqrt{5},2,4)$. Generalize to formula for OP if P has coordinates (x,y,z). Use a rectangular box as a model.

   Consider $P_1(x_1,y_1,z_1)$ and $P_2(x_2,y_2,z_2)$ where P_1 and P_2 are in the first octant. Practice visualizing planes containing the points P_1 and P_2, and the resultant box or rectangular parallelepiped formed. Establish the coordinates of the 8 corners of the box and the lengths of the sides of the box, then determine the length of diagonal.

   Practice finding some distances between points in space.

Have student consider the formulas for the midpoint of a segment in a one-dimensional and two-dimensional coordinate system and then guess the formula for the three-dimensional coordinate system. Some students may want to prove their guess.

Describe geometrically and algebraically the set of points in space 3 inches from the origin.

3.3 Algebraic description of subsets of space.

1. Sentences in one variable.

   half-spaces: $x > k$, $y < k$, $z < k$, etc.

   $\{(x,y,z): x > k\}$

   3-dimensional rectangular region:

   $\{(x,y,z): k \leq x \leq m \text{ and } n \leq y \leq p \text{ and } q \leq z \leq r\}$

2. Discovery exercises.

   Describe the following sets of points:

   (1) (a) $\{x : x = 1\}$
   (b) $\{(x,y) : x = 1\}$
   (c) $\{(x,y,z) : x = 1\}$
   (d) $\{(x,y) : x + y = 1\}$
   (e) $\{(x,y,z) : x + y = 1\}$
   (f) $\{(x,y,z) : x + y + z = 1\}$

   (2) (a) $\{x : x^2 = 1\}$
   (b) $\{(x,y) : x^2 = 1\}$
   (c) $\{(x,y,z) : x^2 = 1\}$
   (d) $\{(x,y) : x^2 + y^2 = 1\}$
   (e) $\{(x,y,z) : x^2 + y^2\}$
   (f) $\{(x,y,z) : x^2 + y^2 + z^2 = 1\}$

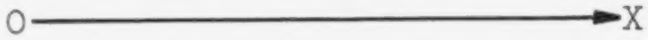
   (3) In all parts of (2) change the symbol "=" to "<".

   (4) In all parts of (2) change the symbol "=" to ">".

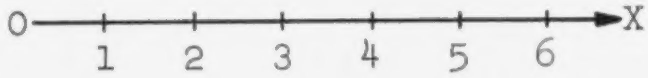
   (5) Describe geometrically and algebraically the set of points in space 3 units from the segment with endpoints A(0,0,0) and B(5,0,0).

Section 4. Polar Coordinate System:

Consider a fixed point 0 and the ray $\overrightarrow{OX}$ in a plane as pictured:

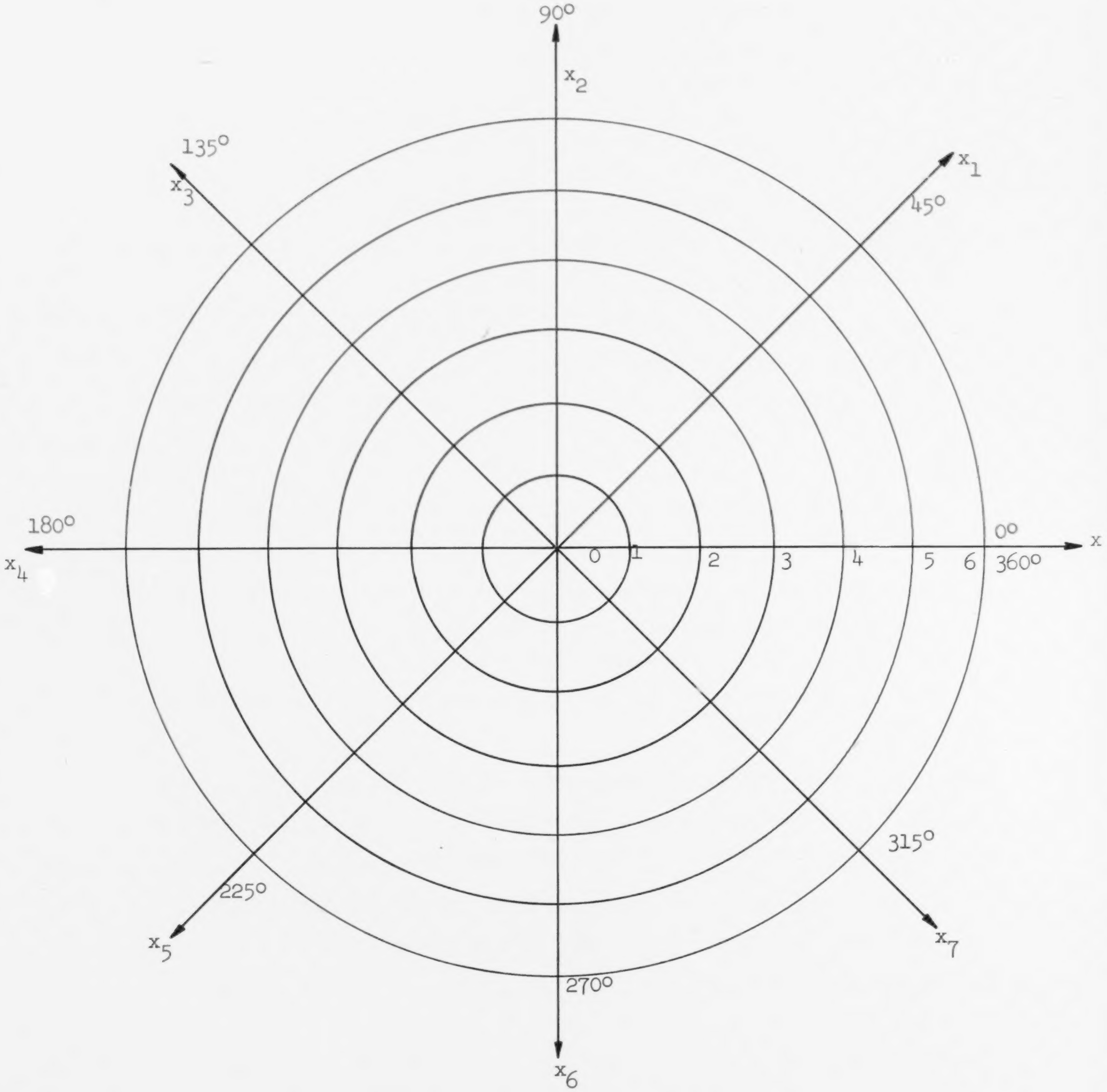


Now put a scale on the ray as pictured:



Imagine the ray $\overrightarrow{OX}$ with the scale on it rotating counter-clockwise around 0 (that is point 0 remains fixed). Imagine the point at 1 tracing out a circle of radius 1, the point at 2 tracing out a circle, radius 2, etc. If you allow the ray $\overrightarrow{OX}$ to rotate counterclockwise through 360° it will have returned to its original position and the whole plane will be covered with concentric circles at 0 and radii 1, 2, 3, 4, etc.

Below is a picture of part of the plane:


90°
x_2
135°
x_3
x_1
45°
180°
x_4
0°
x
0
1
2
3
4
5
6
360°
315°
225°
x_5
x_7
270°
x_6

If ray $\overrightarrow{OX}$ is rotated 45°, then it is shown as ray $\overrightarrow{OX_1}$ in the picture.

Question: Can you locate a point P in the plane if you know that the ray $\overrightarrow{OX}$ will pass through the point for the first time after a counterclockwise rotation of 135° and that the distance from O to P is 3? Try other rotations and any distance from the point O that you wish.

Now consider the other question; pick any point P in the plane; can you find a counterclockwise rotation of $\overrightarrow{OX}$ that will pass through P and also find the distance from O to P?

This is clearly a scheme that assigns to each point in the plane a rotation (angle) and a number. The point O is called the pole and $\overrightarrow{OX}$ is called the polar axis of a polar coordinate system.

Questions:

1. Could you assign different rotations and the same number to the same point?
2. Could you assign different rotations and different numbers to the same point? If a point P can be located by a rotation of s° and a number r (the distance OP) then by convention we say that the ordered pair (r,s°) are polar coordinates of the point P.
3. Is there a one-to-one correspondence between the points in a plane and the set of ordered pairs of real numbers for the first component and rotations for the second component?
4. In the exercises have students plotting points that bring out the ideas discovered in answering the above questions.
5. Describe geometrically the locus of all points (r,s°) in the plane if:

   (a) $r = 3$

   (b) $r < 3$

   (c) $r > 3$

   (d) $2 < r < 3$

   (e) $s^\circ = 90^\circ$

   (f) $s^\circ = 60^\circ$

GRADE 8 - CHAPTER 3

DISPLACEMENTS

Purpose:

To develop a mathematical system using the physical concept of displacement. To introduce the concept of a vector that will be a useful tool in science as well as mathematics. The language of vectors and its use in science is appearing earlier in the student's educational program. The concept of a vector will be useful for all citizens to help them understand the world in which they live.

Background Assumptions:

Knowledge of the structure of the rational number system, rectangular coordinate system in the plane, properties of parallelograms, and the Pythagorean theorem.

Moredock and Sandmann:

The following is an adaptation of portions of the outline by Moredock and Sandmann. They propose much exploratory work using acetate. We heartily endorse the exploratory technique throughout the development of the chapter, but we suggest that more readily available materials may be as satisfactory as acetate. A force table to show the relationships of the mathematical model to a particular physical situation might prove helpful at the proper time.

Section 1. Physical Quantities:

1.1 Quantities

As examples we mention:

length of an object	10	feet
speed of an airplane	550	miles per hour
volume of a test tube	1000	cubic centimeters

These physical quantities can be represented by a segment using an appropriate scale. The quantity is specified by naming a physical unit of measurement and a number. Thus, in the third example above, the unit of measurement is the cc and the number is 1000.

1.2 Operations (review)

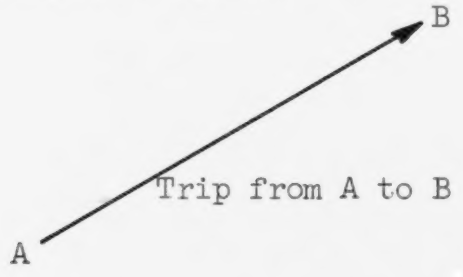
A very brief review of appropriate operations (addition, multiplication) on the above type of physical quantity.

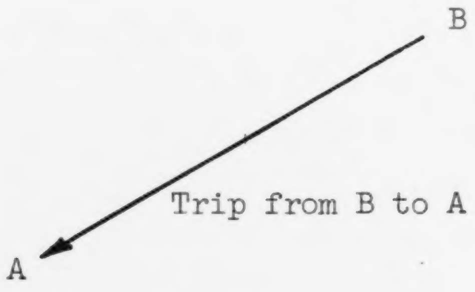
(a) The total mileage traveled on a five-day vacation trip, given the mileages day by day.

(b) The number of cubic centimeters in a gallon is about 3785. How many gallons can be put in a tank measuring 16 meters long, 10 meters wide, and $7\frac{1}{2}$ meters high?

Section 2. Vector Quantities:

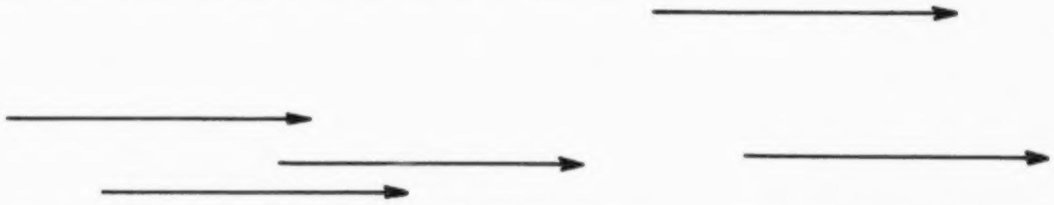
There are quantities that cannot be adequately described by a measurement on a scale alone. Describing a trip along a road involves distance -- that is, a number (referred to a unit of measurement) -- and a direction. Each of the two diagrams below shows the same pair of points A and B.





These arrows (subsets of rays) give two bits of information -- the length of the arrow denotes the distance traveled and the arrowhead indicates the direction. Each arrow has a starting point and an ending point.

These arrows can also be interpreted as showing a displacement, a change in position. In the first case, a body (particle) at A has been moved (or displaced) to B. In the other B is displaced to A; note that it is easy for us to abuse the language by saying that B is displaced to A, when we mean that an object originally at the point B has been moved to the point A.



There are many -- indeed, infinitely many -- two-mile trips in an easterly direction. Observe that the various arrows that represent these trips are all parallel to one another and have the same length. Thus if we model the family of trips by selecting one of these arrows as a representative, we are concentrating on the direction and length aspects.

Now pick out any arrow. See how it is a representative sample of all arrows with the same direction and same length. In fact, it can be used to determine any arrow you wish with that length and direction.

(See Sandmann and Moredock for further details on above.)

Vector is the more common name for the idea represented by the arrow.

We will name the vector by a single letter with an underbar; $\underline{a}$.

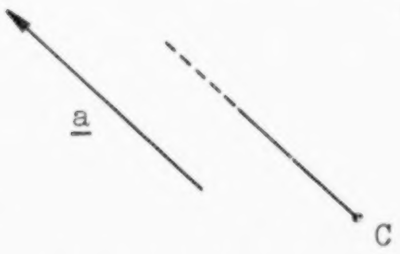
Very brief mention of a few other physical quantities that are vectors: position (bearing), velocity, force -- merely as illustrations that there are further applications they may be studied later.

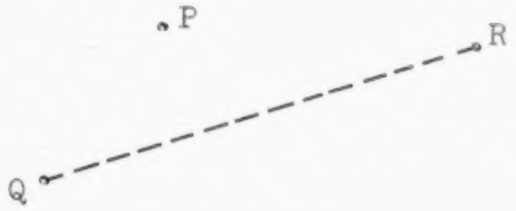
Section 3. Vectors:

An important application of study of vectors is in analyzing changes in the location of a body (displacement). We use this physical situation to motivate most of the succeeding development.

3.1 Activities

In the first diagram below apply the displacement represented by the vector $\underline{a}$ to each of the points C and D. In the second diagram draw the arrow with initial point at P that represents the vector $\underline{b}$ that may be applied to Q to give R.





Activities like this should develop confidence that:

(a) given any point and any vector, the point may be displaced by the vector,

(b) given any ordered pair of points, there is a unique vector that displaces the first point into the second, and

(c) given any point and any vector, there is a unique point which is displaced into the given point by the given vector.

3.2 Equality

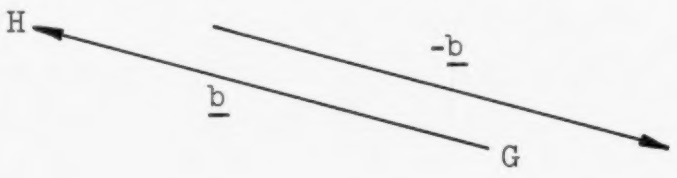
Two concepts need suggesting here:

(a) if vectors are equal, then every point is displaced the same by one vector as by the other;

(b) if just one point is displaced the same by $\underline{a}$ as by $\underline{b}$, then $\underline{a} = \underline{b}$.

3.3 Opposite of a vector

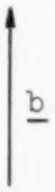
Let $\underline{b}$ be a vector. If G is a point and if G is displaced to H by $\underline{b}$, then there is a vector that moves H into G. This new vector is the opposite of $\underline{b}$ and is denoted by $-\underline{b}$.

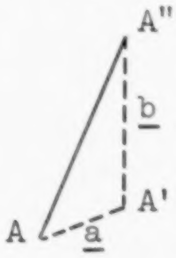


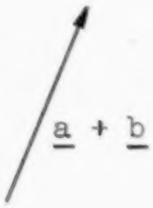
The opposite of a vector essentially backs up, or returns home. Develop confidence in $-(-\underline{a}) = \underline{a}$.

3.4 Addition of vectors

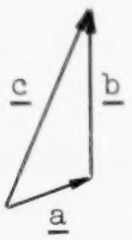
By $\underline{a} + \underline{b}$ applied to A we shall mean the result of applying $\underline{a}$ to A resulting in A' and then applying $\underline{b}$ to A' resulting in A", so $\underline{a} + \underline{b}$ applied to A results in A"; so if $\underline{a} + \underline{b}$ applied to A results in A" and $\underline{c}$ applied to A results in A", then $\underline{a} + \underline{b} = \underline{c}$. This may be pictured as follows:





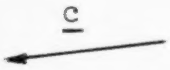


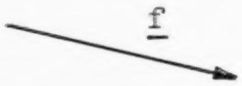
Or it could be pictured this way:



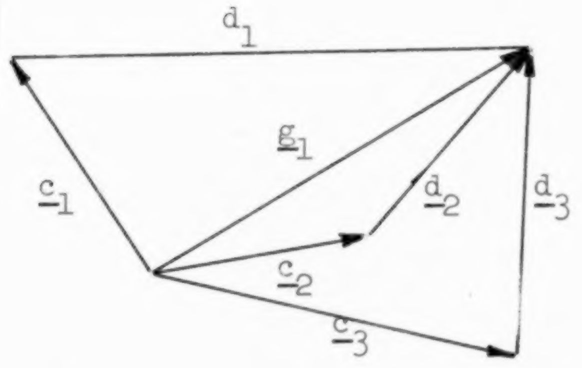
The vector $\underline{c}$ is called the sum (sometimes, resultant) of $\underline{a}$ and $\underline{b}$.

Problem: given vectors $\underline{c}$, $\underline{f}$, find a vector $\underline{e}$ such that $\underline{e} = \underline{c} + \underline{f}$.





Problem: given a vector $\underline{g}$, find two vectors $\underline{c}$ and $\underline{d}$ whose sum is $\underline{c}$. How many such pairs $(\underline{c},\underline{d})$ are there?


d_1
$\underline{g}_1$
$\underline{c}_1$
$\underline{d}_2$
$\underline{d}_3$
$\underline{c}_2$
$\underline{c}_3$

Clearly there are infinitely many pairs.

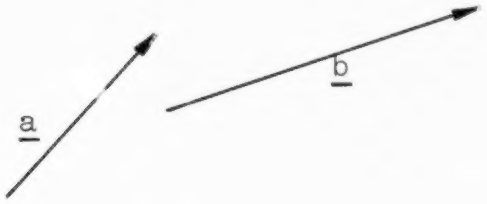
3.5 Zero vector

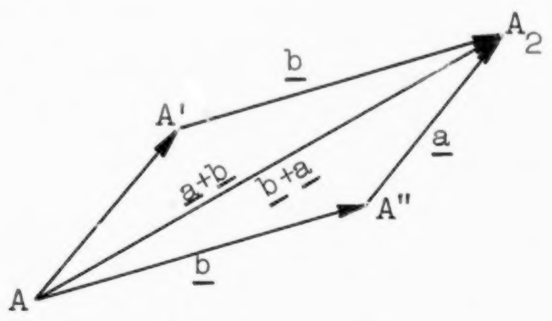
Add a vector and its opposite. Use this to motivate a zero vector. Denote the zero vector by $\underline{0}$. Develop confidence not only in $\underline{b} + (-\underline{b}) = \underline{0}$, but also in $(-\underline{b}) + \underline{b} = \underline{0}$. Also discuss $\underline{c} + \underline{0}$ and $\underline{0} + \underline{c}$.

The zero vector acts, in vector addition, like the number zero in number addition. The opposite of a vector acts, in vector addition, like the opposite of a number in number addition.

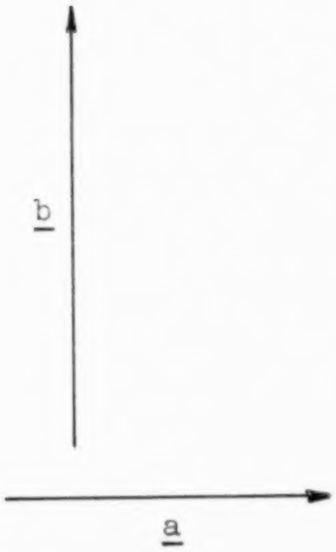
3.6 The commutative principle for vector addition

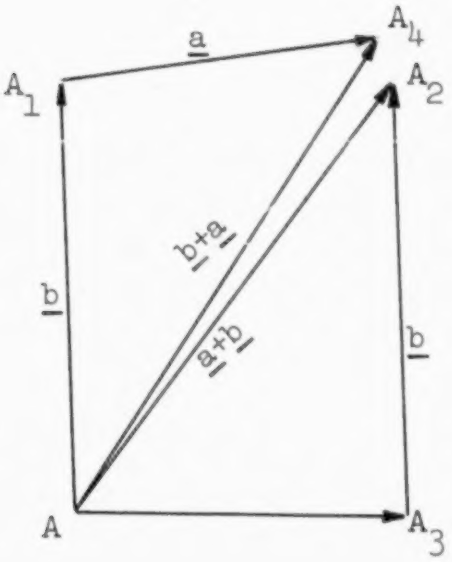
Is $\underline{a} + \underline{b} = \underline{b} + \underline{a}$?


$\underline{a}$
$\underline{b}$


A_2
$\underline{b}$
A'
$\underline{a}$
$\underline{a+b}$
$\underline{b+a}$
A"
$\underline{b}$
A

From the picture it appears that $\underline{a} + \underline{b} = \underline{b} + \underline{a}$. Recall the special cases of $\underline{c} + (-\underline{c}) = (-\underline{c}) + \underline{c}$ and $\underline{c} + \underline{0} = \underline{0} + \underline{c}$. However, in general, maybe it is like the following, with $A_2 \neq A_4$.

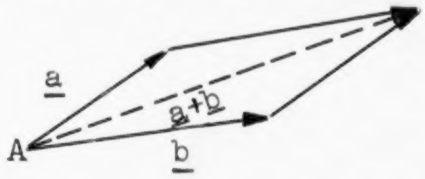




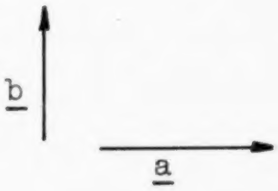
Prove that $A_2 = A_4$, using congruence of triangles, etc.

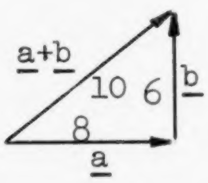
So we see that vector addition is commutative just as addition of real numbers is commutative.

From the picture illustrating $\underline{a} + \underline{b} = \underline{b} + \underline{a}$, we see that one way of finding the displacement of A when $\underline{a}$ and $\underline{b}$ are both applied to A in succession is by "completing the parallelogram".



Using this idea solve the following problem: Forces acting in specified directions on objects may be represented by vectors. Find the resultant of forces F and F' acting on A if F is a force of 8 lbs. acting in an easterly direction and F' is a force of 6 lbs. acting in a northerly direction.

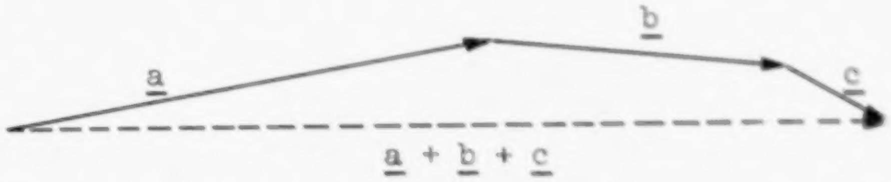




The resultant force on the object is a force of 10 lbs. acting in a direction between east and northeast.

3.7 The associative principle for vector addition.

Consider addition of three vectors $\underline{a}$, $\underline{b}$, $\underline{c}$. Application of $\underline{a} + \underline{b} + \underline{c}$ to a point P means apply $\underline{a}$, then $\underline{b}$, then $\underline{c}$. Develop meaning of sum as vector that "closes the polygon".



Continue discussion to $\underline{a} + \underline{b} + \underline{c} + \underline{d} + \underline{e}$, say. Do not confine attention to the planar case. Develop sum as vector that "closes the polygon", where polygon may be nonplanar.

Can shortcuts in an extended sum be taken? Use this to motivate inquiry about associativity. Show in the usual fashion that $\underline{a} + (\underline{b} + \underline{c}) = (\underline{a} + \underline{b}) + \underline{c}$.

Thus the associative law for vector addition looks much like the associative principle for addition of numbers.

3.8 Summary and review of properties of addition.

$\underline{a} + \underline{b}$ is a vector (closure)
$\underline{a} + \underline{b} = \underline{b} + \underline{a}$ (commutative)
$(\underline{a} + \underline{b}) + \underline{c} = \underline{a} + (\underline{b} + \underline{c})$ (associative)
zero vector $\underline{0}$ exists such that $\underline{a} + \underline{0} = \underline{a} = \underline{0} + \underline{a}$
for each vector $\underline{a}$, opposite $-\underline{a}$ exists such that
$\underline{a} + (-\underline{a}) = \underline{0} = (-\underline{a}) + \underline{a}$.

Relate these properties to those for the addition of numbers. Point out that these properties make the addition of vectors structurally the same as the addition of numbers (integers, rationals, or reals).

This may be the first time that students have seen a mathematical system for something other than numbers, and this should be exploited here.

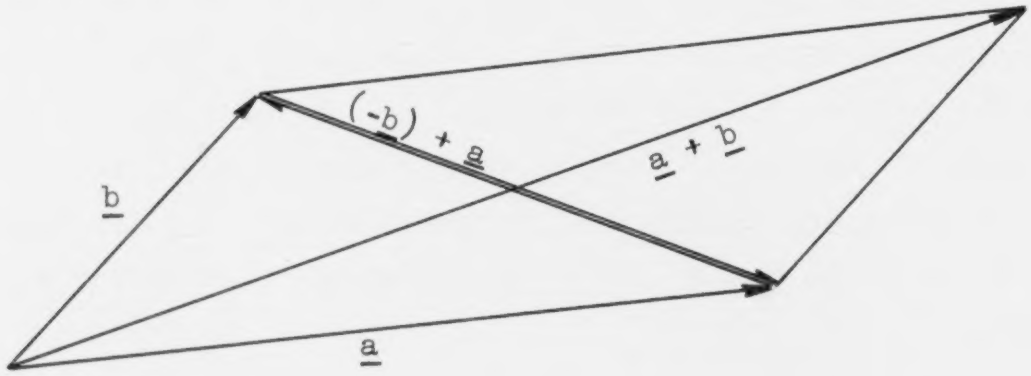
3.9 Solution of vector equations.

Given a point D and a vector $\underline{a}$ and a vector $\underline{b}$. Suppose that D is displaced to A by $\underline{a}$ and that D is displaced to B by $\underline{b}$. Is there a change of position from A to B? Yes, by Section 3.1(b). How can this displacement be expressed in terms of $\underline{a}$ and $\underline{b}$? Observe that we are trying to solve the equation $\underline{a} + \underline{\ ?\ } = \underline{b}$.

One way of describing the displacement from A to B is to take the path from A to D to B. Vectorially expressed, this means that the desired displacement is the sum of $-\underline{a}$ and $\underline{b}$. By associativity, we verify that $(-\underline{a}) + \underline{b}$ satisfies our equation: $\underline{a} + ((-\underline{a}) + \underline{b}) = (\underline{a} + (-\underline{a})) + \underline{b} = \underline{0} + \underline{b} = \underline{b}$.

Note that we are deferring until Grade 9 the discussion of an operation called subtraction. Of course there is no objection if a student realizes for himself that all the ingredients are here and if he wishes as an individual project to pursue the properties of such an operation.

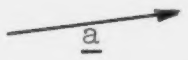
If vectors $\underline{a}$ and $\underline{b}$ are represented by arrows with common starting point, then we may (usually) form a parallelogram with these segments as adjacent sides. One diagonal, with appropriate arrowhead, represents the sum $\underline{a} + \underline{b}$; the other diagonal represents $(-\underline{a}) + \underline{b}$ or $(-\underline{b}) + \underline{a}$, according to the positioning of the arrowhead.

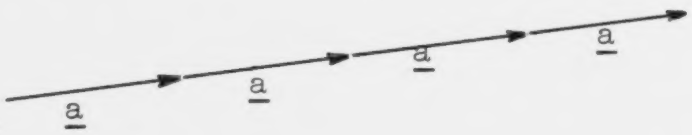


Section 4. Multiplication of a vector by a number:

4.1 Developing meaning of multiplication.

Consider $\underline{a} + \underline{a} + \underline{a} + \underline{a}$.

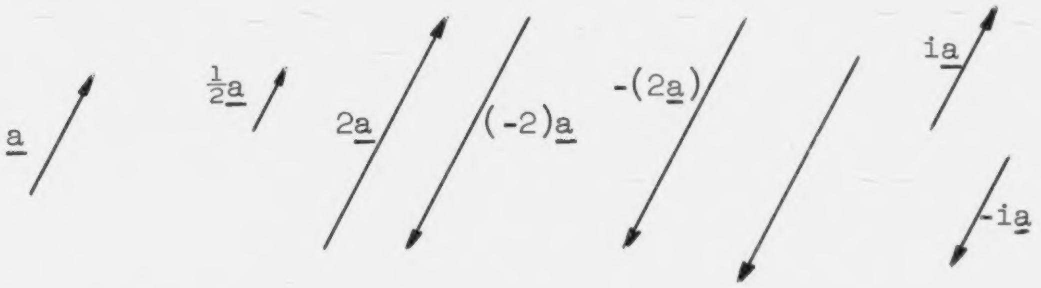




We like to have a name for the vector $\underline{a} + \underline{a} + \underline{a} + \underline{a}$. Let us call it $4\underline{a}$. We note that 4 is a number and $\underline{a}$ is a vector. The vector $4\underline{a}$ is called the product of the number 4 and the vector $\underline{a}$. The operation of forming the product of a number and a vector is called multiplication.

So far in our development the number has been a positive integer. Now develop an intuitive feeling for the meaning of the product of a real number by a vector.

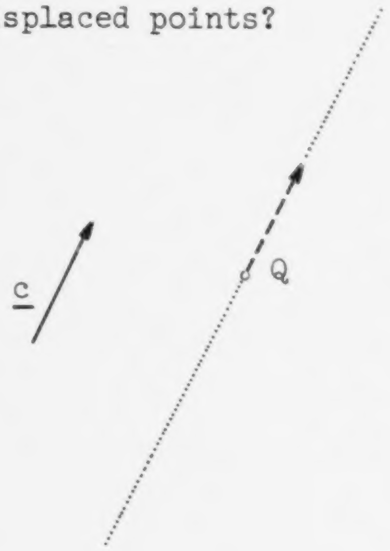
Examples:



In particular, develop intuition that the arrowhead is reversed in case the multiplier is a negative number. Then we are able to observe the following: $(-2)\underline{a} = -(2\underline{a})$. Here $(-2)\underline{a}$ is the product of the number -2 and the vector $\underline{a}$, while $-(2\underline{a})$ is the opposite of the vector obtained by multiplying the number $+2$ by $\underline{a}$. Since the results are the same, we may simplify the notation and write simply $-2\underline{a}$, interpreting it in either way we like.

4.2 Multiplication and parallelism.

Problem 1: Given a vector $\underline{c}$ and a point Q. Consider the various points obtained by displacing Q by the vectors $k\underline{c}$ for all numbers k. What is the locus of these displaced points? After plotting several displacements of Q, the students should observe that for each k the point into which Q is displaced by $k\underline{c}$ lies on the line through Q parallel to an arrow representing $\underline{c}$.



Problem 2: In Problem 1 consider the line through Q that is parallel to an arrow representing $\underline{c}$. Pick any point R on this line. Is there a number k such that $k\underline{c}$ applied to Q will result in R?

Suppose, for example, that the distance QR is twice the length of $\underline{c}$; in this case, $k = 2$. In general we see that there is a number k and that in order to find it we compute QR and divide QR by the length of $\underline{c}$. This is like measuring distance, using the length of $\underline{c}$ as the unit of distance.

Since the above is always possible, we now have a one-to-one correspondence between the points on the line and the real numbers.

Exercises here could be to find endpoints for $K\underline{c}$ applied to Q and to find the numbers k such that $k\underline{c}$ maps one given point into another given point.

Goal is to develop confidence in the following property: two nonzero vectors can be represented by parallel arrows if and only if each of the vectors is a multiple of the other by a number.

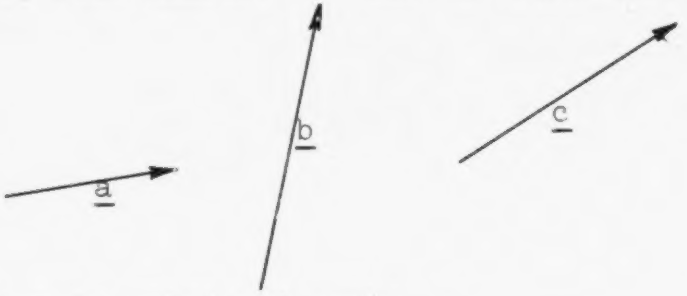
Section 5. Translation:

A vector describes a translation of the plane (or of space) WST.

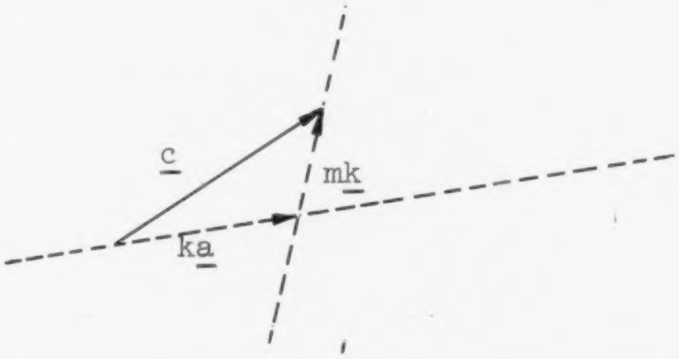
Section 6. Decomposition:

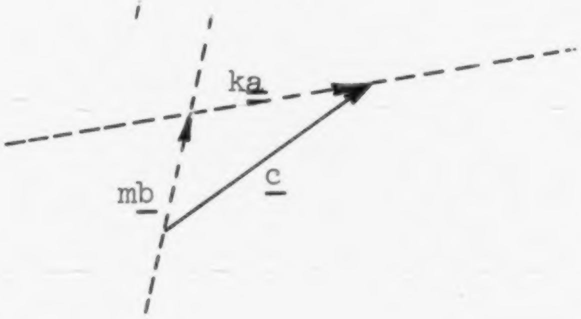
6.1 Decomposing a vector in terms of two given vectors.

Problem: Let $\underline{a}$ and $\underline{b}$ be two (noncollinear) vectors and let $\underline{c}$ be any vector. Express $\underline{c}$ in terms of $\underline{a}$ and $\underline{b}$, that is, find numbers k and m such that $\underline{c} = k\underline{a} + m\underline{b}$.



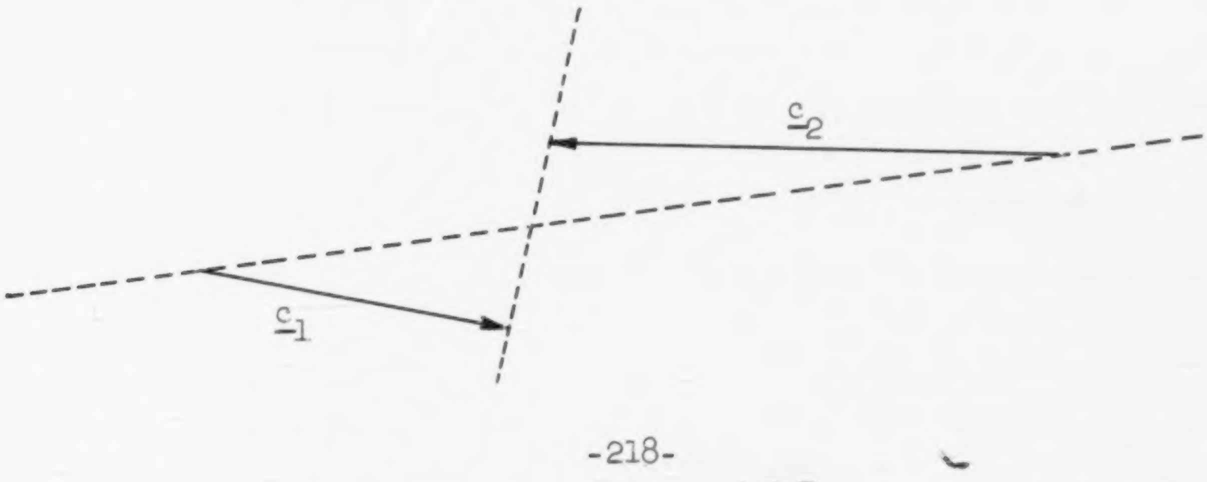
Let P be a point and apply $\underline{c}$ to P obtaining Q. Draw the line through P parallel to $\underline{a}$ and the line through Q parallel to $\underline{b}$ and note the intersection of the lines. We may measure k and m so that $\underline{c} = k\underline{a} + m\underline{b}$.





Observe that the same result is obtained by drawing the line through Q parallel to $\underline{a}$ and the line through P parallel to $\underline{b}$. This is an application of the commutative property for vector addition.

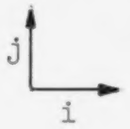
For other problems, retain the same $\underline{a}$ and $\underline{b}$, but alter $\underline{c}$ so that we need a negative number for k and/or m.



6.2 Naming vectors.

From the previous work it should be clear that any vector can be expressed in terms of two nonzero nonparallel vectors.

Ask the students to think about two convenient vectors such that all other vectors can be expressed in terms of these two in the above manner. Eventually show them two unit vectors that are perpendicular, and label them $\underline{i}$ and $\underline{j}$.



Now consider $m\underline{i}$ for all numbers m, $n\underline{j}$ for all numbers n, and $m\underline{i} + n\underline{j}$ for all numbers m, n. Recognize as the familiar rectangular coordinate system.

Typical Exercises:

Plot $4\underline{i} + 2\underline{j}$, $-3i + (\frac{5}{3})\underline{j}$. Given vector $\underline{c}$ in plane of $\underline{i}$ and $\underline{j}$, find the coefficients of $\underline{i}$ and $\underline{j}$ (by graphical means) in the decomposition of $\underline{c}$.

Section 7. Extension to Vectors in 3-space:

With three mutually perpendicular unit vectors $\underline{i}$, $\underline{j}$, $\underline{k}$, extend the development in Section 6 to three-space. This should not be hit too hard, but it will be helpful in developing spatial visualization and space perception.

It should be clear that $p\underline{i} + q\underline{j} + r\underline{k}$ for all real numbers p, q, r gives the three-dimensional rectangular coordinate system.

Section 8. Applications:

If applications to the "real world", world of work, etc., can be found that are understandable by the 8th grader, they should be included in this chapter. Possible topics are vector diagram for air speed, ground speed, and wind velocity for an airplane in flight; same for a ship steaming in the ocean with a current.

GRADE 8 - CHAPTER 4

PROBLEM ANALYSIS (STRATEGIES)

Background Assumptions:

Graphing in one and two dimensions.

Informal introduction to use of variable as a symbol for a number.

Techniques for finding solutions of sentences in one and two variables.

Real numbers and properties.

Rational:

Students should develop many valid patterns for problem-analysis, such as: willingness to guess solutions and then derive further information from the resulting data, working backward on a problem, sketching (pictures), organization of data in tables, graphing, recognition that a functional relation is involved; in short, willingness to approach problems in more than one way in order to organize the final presentation of the problem solution.

Purposes:

1. To provide the student with the techniques which will enable him to translate the conditions of a problem into a mathematical sentence or system of sentences.
2. To provide the student with a variety of techniques for problem analysis, and develop flexibility in his manner of approach.

UNDER NO CIRCUMSTANCES SHOULD A STUDENT BE FORCED TO GO THROUGH A FORMAL PROCEDURE OF ANALYSIS IF HE HAS DISCOVERED A METHOD OF SOLVING A PROBLEM EARLY IN THE ANALYSIS.

Note: In the writers opinion, and others, Sections 1 and 2 are far too "heavy". Much of what is in these sections could occur in the Teacher's Commentary as directions for the Teacher.

How do people make discoveries? What is the process (if any) that leads to the solution of problems in mathematics, nuclear fission, rocket design, neural surgery, conservation of natural resources, etc? These are possibly questions that many students might ask. However, many of them really want to know how they can invent correct solutions to problems and why these solutions are constructed the way that they are. The purpose of this chapter is to give you some help in organizing your approach to problem solving in mathematics and to, perhaps, improve your problem solving techniques.

Section 1. Translation of Phrases:

1.1 Mathematical Phrases to English Phrases.

Many problems that occur are stated either orally, in written form, or in both forms. They usually are not too clearly understood at first, and they usually stimulate more questions than can be answered immediately. (In fact, one significant activity in problem analysis is very often trying to determine what the real questions or concerns are in a problem.) However, one first major step toward the solution of some problems is one of translation of the problem into a form which will permit some form of organized analysis. The English language is far too disorganized to permit, in general, any efficient problem solving techniques to be developed. For example, read the following excerpt from an insurance policy:

> premises means unless otherwise indicated (1) all premises where the named insured or his spouse maintains a residence and includes private approaches thereto and other premises and private approaches thereto for use in connection with said residence, except business property and farms, (2) individual or family cemetery plots or burial vaults, (3) premises in which an insured is temporarily residing, if not

> owned by an insured, and (4) vacant land, other than farm land owned by or rented to an insured. Land shall not be deemed vacant following the commencement of any construction operations thereon unless such operations are being performed solely by independent contractors in connection with the construction of a one or two family dwelling for the insured.

As you can see this statement, while useful, is very cumbersome to apply to a situation. In mathematics the attempt is consistently made to keep definitions, statements, etc., as clear and concise as possible.

In order to effectively develop your problem solving skills and understandings, we will first look closely at the process of translation. In the beginning many of the phrases you will be asked to translate will be simple. It is extremely important that you practice carefully the techniques introduced so that you can develop some skill in working with more complex situations.

1.2 Class Discussion.

Try to write an English phrase which clearly interprets the mathematical phrase:

$$R - \frac{1}{4} W$$

If you had some difficulty in doing the above, relax; you are probably not alone.

Let's see what we do know and what we don't know about this phrase. You should remember that variables such as R and W are usually names for numbers. You do not know exactly what number they represent, but you do know that they represent definite but unstated numbers. Furthermore, you should notice that the phrase represents the DIFFERENCE of the two numbers R and $\frac{1}{4} W$, and that the number $\frac{1}{4} W$ is the result of multiplying the numbers $\frac{1}{4}$ and W.

Now you are faced with the task of finding out something more about the variables. You know that R represents a number, BUT is it the *number of rings around Saturn*? *Is it the number of rocks in a garden*? *Is it the number of redheads in a mathematics class*?

As you can see, there are unlimited possibilities for R, and it is impossible to state exactly what R or W could represent until we know more about the origin of the phrase. Furthermore, it becomes obvious that you must be able to state *clearly* what each variable in a phrase represents before you make your first attempt to translate or interpret the mathematical phrase. (Note: you should not hesitate to revise your statements, throughout the translation process; in fact, this process is encouraged as you gain more understanding about the situation.)

For example, you could say: "Let R represent the answers on a test".

First revision: "Let R represent the number of answers on a test."

Second revision: "Let R represent the number of correct answers on a test. And let W represent the number of wrong answers on a test".

The point here is that you should try to identify the variables as clearly and completely as possible (revising many times, if necessary). Now one possible translation of the above phrase might be the following:

> If R represents the number of correct answers on a test, and W represents the *number* of wrong answers on the same test, then the phrase $R - \frac{1}{4}W$ is: "the difference between the number of correct answers on a test and one-fourth of the number of wrong answers".

1.3 Some Characteristics of a Translation of a Mathematical Phrase to an English Phrase Are:

(1) The variables are clearly identified (usually as numbers).

(2) The variables are as completely identified as possible. (Revisions are encouraged, if necessary.)

(3) Basic operations are identified and considered in the translation.

(4) The context of the translation of the phrase makes sense, and the final translation is a correct English phrase.

1.4 Exercises:

Write an English phrase which is correct translation of the given mathematical phrase. Be sure to state clearly what each variable represents in your translation.

1. $60t$
2. $2L + 2W$
3. $(3.14)d$
4. $\frac{1}{2}bh$
5. s^2
6. prt
7. $n + 8$
8. $\frac{x + y}{2}$
9. $a^2 + b^2$
10. $\frac{d}{r}$
11. $(2)(3.14159)r$
12. $lw + lw$
13. $a + b + c$
14. $x = 2y$
15. $m - 5$
16. $(186{,}000)^2\ m$
17. $\frac{4}{3}(3.14)\ r^3$
18. $s + (s - a) + 7$

1.5 Translation: English Phrases to Mathematical Phrases:

Do you remember that the symbols:

(1) "+" in a phrase is sometimes translated "the sum of", or "exceeds". (Name some other translations.)

(2) "-" in a phrase is sometimes translated "less than", "the difference between", "decreased by", etc.

(3) " $\cdot$ " in a mathematical phrase is sometimes translated "the product of", "times", etc.

(4) "$\div$" in a mathematical phrase is sometimes translated "the quotient of", "divided by", etc.

"Translation" is used here in the loose sense meaning that these English words sometimes lead to the indicated operations of $+, -, \times, \div$.

1.6 Classroom Discussion:

There are many English translations which can represent the symbols indicating the operations of addition, subtraction, multiplication, and division. Can you write two more English translations of the above symbols?

1.7 Characteristics of Translations of English Phrases to Mathematical Phrases:

In translating an English phrase to a mathematical phrase it is important that you practice:

(1) Identifying clearly all variables which are used (as numbers).

(2) Identifying as completely as possible all variables used. (It becomes important in communicating your ideas to yourself and others that you try to use complete, correct, English sentences to do this.)

(3) Recognizing the English translations of symbols of operations in all of their various forms.

1.8 Exercises:

In each of the exercises write a mathematical phrase which is a translation of the English phrase. Identify clearly what the variable or variables represent. Choose a variable if none is given, and use only one variable unless directed to do otherwise.

1. The number of dollars earned in t hours at three dollars an hour.
2. The total number of yards in k feet of cotton material and n yards of wool material.
3. The difference in cents between q quarters and d dimes.
4. The number of yards in t feet.
5. The average of three test scores x, y, and z.
6. The sum of two consecutive integers.
7. The product of two consecutive even integers.
8. Fifteen inches more than twice the number of inches in the length of a rectangle.
9. One thousand times the thrust of a Saturn rocket.
10. The number of dollars in the cost of a house increased by fifteen percent of the cost of the house.
11. The number of miles traveled in t hours at 6000 mph.
12. The cube of a number.
13. The square of a number decreased by the square of another number.
14. Seven more than a number.
15. The sum of a number and its reciprocal.
16. Thirty percent of x pounds of gold.
17. The product of two numbers increased by the first number. (Use two variables.)
18. The sum of the squares of the digit of a two-digit number. (Use two variables.)
19. The larger of two numbers multiplied by the difference of two numbers. (Use two variables.)
20. The square of an integer diminished by the difference of the product of the integer and the next consecutive integer.

21. The area of a triangle increased by fifteen.

22. The difference between the numerator and the denominator of a fraction if the numerator exceeds the denominator by 5.

23. The sum of the reciprocals of a number and a larger number. (Use two variables.)

24. The tens digit of a two digit number is three more than the units digit. Write a phrase representing the number. (Use only one variable.)

25. Write a phrase for the number of inches in the perimeter of a square whose side is s inches long.

26. The ratio of calories in the soft drink "Instant Pop" to "Brand X" is one to nine. Write a phrase for the number of calories in a bottle of "Brand X" in terms of the number of calories in a bottle of "Instant Pop".

27. The speed of a particular satellite in orbit decreases by 6 miles per day. Write a phrase representing the number of miles decrease in speed over a period of 36 hours.

28. The number of students receiving A's in one mathematics class is determined by squaring the number of students in the class, decreasing the resulting number by the product of the square root of three and the number of chairs in the room, and then dividing this quotient by 295. Write a phrase representing the number of students receiving A's in a mathematics class.

29. Write a mathematical phrase representing the speed of a plane flying with the jet stream if the plane is moving at a constant speed of 500 miles per hour.

30. Write a mathematical phrase representing the speed of a plane flying against the jet stream which is moving at a constant speed of 500 miles per hour.

31. Write a mathematical phrase representing the number of miles traveled by a plane flying for 6 hours with the jet stream which is moving at a constant speed of 500 mph.

32. Write a mathematical phrase representing the number of miles traveled by a plane flying for 6 hours against the jet stream which is moving at a constant speed of 500 mph.

33. The temperature now, decreased by 32.

34. The product of the temperature now, decreased by 32, and $\frac{5}{9}$.

35. The product of a number and the sum of two numbers. (Use two variables.)

36. A number increased by the sum of two numbers. (Use two variables.)

37. The product of the first number and the second number increased by the product of the first number and the third number.

38. Given three numbers a, b, c the opposite of the second number increased by the square root of the difference between the square of the second number and the product of four, the first number and the third number all divided by the product of two and the first number.

39. The square of the first number diminished by twice the product of the first number and second number, increased by the square of the second number. (Use two variables.)

40. The product of the sum of two numbers and the difference of the same two numbers. (Use two variables.)

Section 2. Translation of Sentences:

2.1 Mathematical Sentences to English Sentences.

You have been using mathematical sentences formed by combining mathematical phrases such as "3n + 1", "2", "25 - 7", etc., with mathematical verb forms such as "=", ">", "≤", etc. Many times one or two mathematical sentences represent, or serve as a model of a situation, which takes several English sentences to describe.

Consider the following sentence; $35x + (70)(40) = 50(x + 40)$. Translating this sentence is like writing a story when you know the ending. (I.e., for now, the ending to our story is the sentence: $35x + (70)(40) = 50(x + 40)$). Let's see if we can work backward and write a problem which is an English translation of this sentence. This process should provide you with some insight into how to go about translating and analyzing some problems.

2.2 Class Discussion:

$$35x + (70)(40) = 50(x + 40)$$

(1) Can you tell what situation or situations this particular sentence represents?

(2) What do you need to know first so that all of you will interpret this sentence in a similar manner?

(3) You know that the variable, x, must represent a number but the number of what? Could it represent:

  (a) The number of pounds of candy selling for 35 cents a pound?

  (b) The number of ounces of uranium in a radioactive compound of uranium and radium?

  (c) The number of hours a car is driven at an average speed of 35 miles per hour?

(4) Write another phrase describing the variable x.

2.3 The translation Process (still classroom discussion):

Let's agree to the following statement as one interpretation of the variable x.

"Let x represent the number of ounces of gold in a compound." (It is only fair to point out that we might have to revise and improve this statement as we become more involved in the problem.)

Now:

(1) What does 35x represent? You can see that we also have to know what 35 represents. Let 35 be the cost in dollars of one ounce of gold.

(2) Now state what 35x represents. Is the following statement correct? "35x represents the cost in dollars of x ounces of gold."

(3) What does 40 represent? Could it represent the following? "40 represents the number of ounces of pure platinum in the compound". (Note that you now have decided that your compound consists of gold and platinum. Revise your statement about the variable.)

(4) What can 70 and (70)(40) represent now? "70 represents the cost in dollars of one ounce of pure platinum". Why did we choose one ounce? Would you believe 2 ounces? "(70)(40) can represent the cost in dollars of 40 ounces of platinum at 70 dollars an ounce".

(5) "=" means "is equal to", "is the same as", etc.

(6) "x + 40" now can represent the number of ounces of gold added to the number of ounces of platinum". Do you see why we can say this now, but could not state it at the begining?

(7) Now "50" can represent the cost in dollars per ounce of the mixture of gold and platinum.

The following is one possible translation:

> The plating of a secret satellite must be a mixture of pure platinum and gold. Exactly forty ounces of pure platinum must be used in order for the satellite to perform correctly. Pure platinum costs 70 dollars an ounce, and gold costs 35 dollars an ounce. The cost of the mixture of platinum and gold must be 50 dollars an ounce. How many ounces of gold must be used in order to make a mixture which costs 50 dollars an ounce?

Now try writing a translation of your own where x represents the number of pounds of candy selling for 35 cents a pound.

2.4 Some Characteristics of Translations of Mathematical Sentences to English Sentences:

(1) The variable or variables are clearly identified as to what they represent. (Usually a number of ...)

(2) Each part of the sentence is clearly identified as to what it represents.

(3) The translation must make sense. In other words, the parts must fit together in a reasonable way (i.e., x cannot represent the number of tickets sold for a car-wash in the same problem).

(4) You must be willing to *write*, and *think*. The authors have never heard of a single student being permanently injured by "writers cramps" or "over-thinking", but they have observed hundreds of students succeeding in and enjoying mathematics because they were willing to *read*, *write*, *rewrite*, *think*, *experiment*, and *conjecture*.

2.5 Translation of Sentences Involving Restricted Domains:

Sometimes the domain of the variable is dictated by the given mathematical sentence.

Example: Write an English translation for the following sentence:

$$2x + 1 = 12 .$$

Since the solution set of this sentence is $(\frac{11}{2})$, it would not be meaningful to write an English translation which requires that the variable represent a whole number.

Class problem: Write a meaningful translation of this sentence.

2.6 Exercises:

Write an English translation of the following sentences. State clearly your choice as to what each variable or variables represent, and then state what each part of the sentence represents. Be sure your translation makes sense! Be creative, use your imagination!

1. $24 = 12w$
2. $40 = 2w + 2(3w)$
3. $312 = (\frac{1}{2})(24)(h)$
4. $A = (3.1416)(16)^2$
5. $64 = a^2 + 25$
6. $m - 1 = 17$
7. $2n - 1 = 18$
8. $81 = s^2$
9. $96 = bh$
10. $150 = (1500)(.05)(t)$
11. $763 = \frac{1}{2}(15)(20 + b^1)$
12. $V = (5)(10)(12)$
13. $X - 600 = 100$
14. $X + 600 = 700$
15. $2600 = (r)(4)$
16. $x(3 + a) = 3x + ax$
17. $25q + 10d + 5n = 250$
18. $C + .05C = 2500$
19. $\frac{1}{10} + \frac{1}{20} = \frac{1}{x}$
20. $F = \frac{5}{9}(C - 32)$
21. $x + (x + 1) + (x + 2) = 72$
22. $x > 3$ and $x < 7$
23. $|y| = 5$
24. $m > 2$ or $m < -4$
25. $\frac{x + y + z}{3} = 77.5$
26. $.30x + (.40)(x + 1) = (.50)(17)$
27. $K = \frac{m\ v^2}{(2)(32)}$
28. $E = (.009)v^2$
29. $y = \frac{1}{2}x + 3$
30. $360 = \frac{1}{3}(B)(10)$

2.7 Translation: English Sentences to Mathematical Sentences:

Let's use your ability to translate English phrases to mathematical phrases to write translations from English sentences to mathematical sentences. Remember, in the last section, many times it took only one sentence to represent several English statements. As in the last section, before the final translation of English statements can be made, you must:

(1) know and state clearly what the variable or variables represent;

(2) translate each part of the problem in terms of what the variable represents;

(3) be sure your translation makes sense;

(4) know what the domain of the variable is.

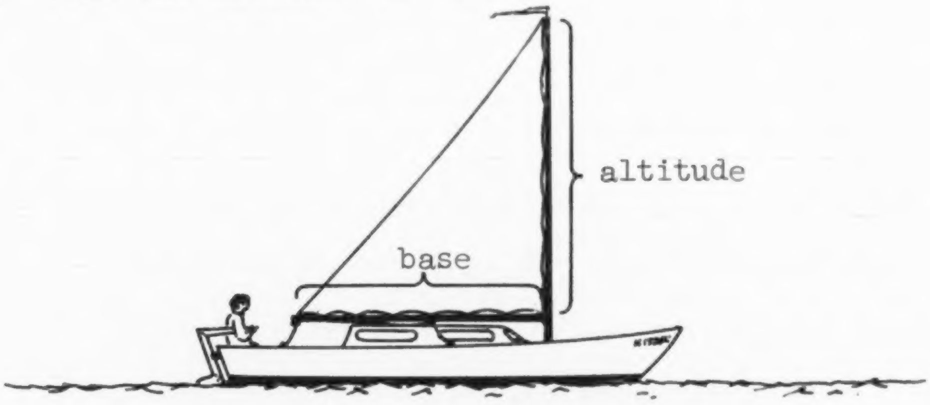
2.8 Classroom Discussion: Try to Complete the Following:

Consider the following problem: (The numerals in parentheses refer to the statements above.)

(2) A triangular sail has a base which is 6 feet less than the altitude.

(3) The area of the sail is 312 square feet.

(1) Find the altitude of the sail.



Your translation might go something like this:

Select a variable or variables and tell what they represent. Let the variable represent something in the problem that you are trying to find. For example:

(1) Let h be the number of feet in the altitude of the triangular sail.

(2) Then h - ? is the number of feet in the base of the sail.

(3) The number of square feet in the area of the sail is?

(4) I know that the area of any triangle is represented by the formula $A = \frac{1}{2}bh$ where A, b, and h are any numbers of arithmetic.

(5) Therefore, the translation of these English sentences into an open sentence is:

$$312 = \frac{1}{2}(?)(?).$$

2.9 Exercises:

Translate the following English sentences into mathematical sentences which could help solve the problems. State clearly what each variable represents and what each part of your sentence represents. Use only one variable unless directed to do otherwise. Do not find the solution sets of the mathematical sentences at this time.

1. The area of a rectangular-shaped lot is 5000 square feet. The length of the lot is twice as long as the width. Find the width of the lot.
2. A first number is three more than a second number. The sum of the two numbers is 31. What are the numbers.?
3. Jimmy is now twice as old as his sister Kathy. In two years the sum of their ages will be thirteen. What are their ages now?
4. A car travels for eight hours at an average speed of 50 miles per hour. How far does the car travel?
5. A car travels 232 miles in 4 hours. What was the average speed of the car?

6. An odd number which is increased by the next consecutive odd number is equal to 148. What are the two numbers?

7. Shady Hills High School collected 175 dollars in one day for a student scholarship fund. The number of nickels was $2\frac{1}{2}$ times the number of quarters and the number of dimes was 1100 more than the number of nickels. How many nickels, dimes, and quarters were collected? (Remember, 175 dollars is the same as 17,500 cents.)

8. The sum of two numbers is 135. The larger number is 23 more than three times the smaller number. What are the two numbers?

9. The number of centimeters in one inch is approximately 2.54. Find the approximate length of a bar of silver, in centimeters, if the bar is one yard long.

10. The number of feet in a mile is 5280. How many miles does a runner go if he runs 1320 yards?

11. The number of square inches in a square foot is 144. How many square feet are there in $7\frac{1}{2}$ square inches?

12. The number of minutes in one hour is 60. A car is traveling at 30 miles per hour. How many miles per minute is the car traveling?

13. The number of seconds in one minute is 60. A car is traveling at 30 miles per hour. How many miles per second is the car traveling?

14. The number of feet in one mile is 5280. A car is traveling 30 miles per hour. What is the speed of the car in:
    (a) feet per hour?
    (b) feet per minute?
    (c) feet per second?

15. A boy runs against a head wind for one mile and averages 10 miles per hour. He returns one mile running with the wind and averages 15 miles per hour. How fast can the boy run in still air (no wind), and what was the speed of the wind? (Use two variables; write two sentences. Remember: $d = rt$.)

16. Students in the mathematics club at Shady Hills High School sold special plastic bookcovers to raise money for their annual vacation in Hawaii. The club received 4,000 dollars from the sale of the bookcovers. What was the cost of all of the bookcovers if the margin of profit was 500 percent?

17. Captain Horatio Bilgewater carries a cargo of snarfs on his ship. Captain Hook carries a cargo of bminfs on his ship. Together they have 33 snarfs and bminfs. The number of snarfs that Captain Bilgewater has is $2\frac{2}{3}$ the number of bminfs that Captain Hook has. How many bminfs does Captain Hook have?

18. Three consecutive odd integers add up to 75. What is the smallest of these three numbers.

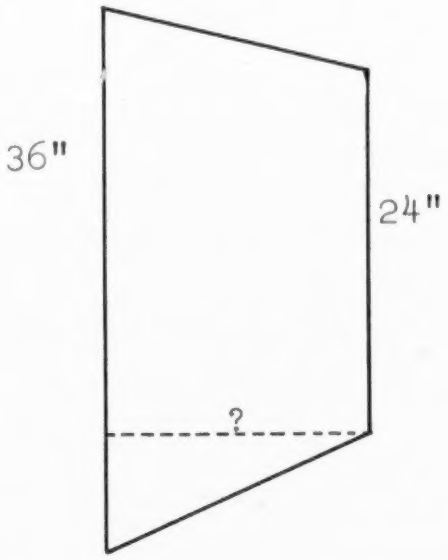
19. Three consecutive odd integers add up to 57. What is the middle number of these three numbers?

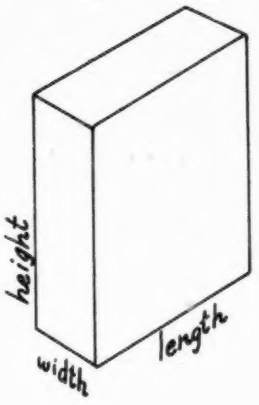
20. Three consecutive odd integers add to 117. What is the largest of these three numbers.

21. What number divided by $\frac{1}{3}$ of itself is equal to 27?

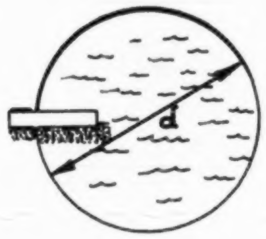
22. Oliver Baconfat, a 300 pound sprinter on the Shady Hills High School track team, wants to buy a pair of size 17 track shoes. The shoes cost 2 dollars more than two times the amount of money Oliver has now. If the shoes cost 20 dollars and 50 cents, how much money does Oliver have now?

23. The area of a trapezoid is 512 square inches. The length of each base is 24 inches and 36 inches, respectively. What is the length of the altitude?





24. The volume of a box is 632 cubic inches. The length is 3 times the width and the height is 4 times the width. What is the width of the box?



25. Horatio Algae, a cousin of Captain Bilgewater's, built a circular swimming pool in his backyard. The pool covers an area of 1700 square feet. What is the diameter of the pool?

26. As a boy, Horatio Algae sold papers on a street corner. He received 1 cent for each paper sold during the week and 2 cents for each paper sold on Sunday. During one week he sold 1700 papers including Sunday sales. He received 22 dollars for the week. How many papers did he sell on Sunday? (Use two variables; write two open sentences.)

27. Horatio Lox, another cousin of Captain Bilgewater's, is a rocket fuel expert and was preparing a new mixture for an experimental missile. He finds that the amount of liquid hydrogen must be exactly 17 percent of the rest of the secret ingredients. The total smount of the final mixture has to be exactly 4 gallons. How much liquid hydrogen has to be used?

28. If you take one-third of a number, you get the same result as if you subtract 93 from the number and add sixteen to one-half of that difference. Find the number.

Section 3. Problem Analysis and Strategies:

3.1 Basic Attitudes Toward Problem Analysis:

There is no permanent procedure or formula for analyzing and solving every problem. It is clear that the reason for analyzing problem situations is to arrive at a solution, if solutions exist.

Now that you have had some practice translating problem situations, let's look more closely at the strategies for problem analysis and try to develop some usuable problem analysis methods. Several techniques will be discussed. Many analysis methods you will eventually use will probably be a mixture of the procedures developed here. Do not hesitate to vary your method of attack.

Basic Principle: Every person can do something toward solving a given problem.

You have developed some ability to translate or create a mathematical model of a stated problem situation. Not all models have to be in the form of mathematical sentences. You could also use drawings, tables of data, graphs, or any combination of these which will organize your understanding of the problem. The most important thing in learning to solve problems is to write or do something to organize the information in as many ways as you can until you score a "break-through". Don't give up!

3.2 Organization Techniques (A First Strategy)

(1) Read or listen to the problem as often as you can. List any words or expressions that you don't understand. Get these cleared up!

If you can solve the problem now, or at any other point in this procedure, then do so! Use an analysis procedure only to explain your work, and, perhaps, to check your result.

(2) List all of the information which is given. You might use translation techniques to do this, tables, graphs, drawings with information indicated on the sketch, or a combination of these procedures. Revise this part continuously as you analyze your problem. Try to decide if any information is given which is not necessary for the solution of the problem. Keep checking on this situation throughout the whole analysis procedure.

(3) Try to state exactly what you're looking for and if possible in what form you think the solution might be. (A number, a graph, a drawing, a table of data, etc.)

(4) Identify specifically anything else that is not known in the problem. You may not be given enough information to solve the problem. This is a good place to start checking on this possibility.

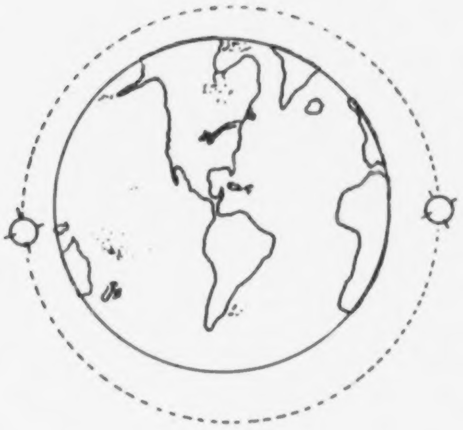
(5) If it is possible to represent the unknown or unknowns by variables, then do so and translate the appropriate parts clearly into mathematical phrases.

(6) Perhaps there is a basic relationship that exists that can be represented by a formula. If so, write it down. If appropriate, write a mathematical sentence or sentences which translates or is a model of the problem. If you have had to simplify the problem, or have ignored any physical properties of the situation in order to write your translation, be sure to indicate this. (Don't be afraid to do this, since it may be the only way to get to a final solution.)

3.3 Example of this Strategy at Work (Class Discussion Problem):

Now let's try these steps on a problem. This problem was chosen particularly, because the analysis and solution are not immediately obvious. We hope to illustrate some things in problem analysis which every student can do! You are not expected at this time to be able to perform the manipulations necessary to determine the answer to this problem, but you can analyze it!

"Two satellites are placed in the same sized circular orbit. The first satellite is traveling 8 miles per minute faster than the second satellite. The faster satellite requires 2 minutes less time for the 30,000 mile trip around the earth than the slower satellite. Find the rate of speed of each satellite in miles per minute."



The Analysis

Step (1) Read the problem.

Step (2) List the given information: (try to use your own words); make a sketch.

(a) The orbit is ______ in form.

(b) Speed of satellite number one is 8 miles per minute ________ than the speed of the second satellite.

(c) Orbit time of first satellite is ________ than the second satellite.

(d) Length of one orbit is ______ for ______ satellites.

Step (3) State exactly what is to be found.

(a) The speed of the first satellite in miles per minute.

(b) The ______ of the second satellite in ______ per minute.

Step (4) What else is unknown?

(a) The time it takes the first satellite to complete the 30,000 mile orbit.

(b) The ______ it takes the ______ to complete the 30,000 mile orbit.

Step (5) Represent the unknowns by a variable or variables, and translate English phrases to mathematical phrases if you can.

(a) Let x represent the speed of the first satellite in miles per minute.

(b) Then ______ represents the speed of the second satellite in miles per minute.

(c) Let t represent the number of minutes it takes the first satellite to complete orbit.

(d) Then _____ represents the number of minutes the second satellite takes to complete the orbit.

Step (6) Write a sentence or sentences which are translations of the problem or represent a basic relationship between the given information and the variables.

(a) The basic relationship that exists is:

distance = (rate)(time)

(b) Therefore for the first satellite, we can write the sentence:

$$30{,}000 = (x)(t)$$

(c) For the second satellite we write:

$$30{,}000 = (x - 8)(?)$$

3.4 A Second Strategy:

If you cannot do Step 6 or any of the other steps, ask yourself these questions and try to answer them:

(1) Have I ever solved a problem like this one before? If so, how?

(2) Is any part of this problem like a problem I have solved before? If so, can I use the procedures I know to solve part of this problem? Can I change the method I know slightly and solve this problem?

(3) Can I draw another figure or figures which will represent all or any part of the problem? If so, do it! At least try drawing several figures and labeling them.

(4) What are some ways I know of that might get an answer like the one I want?

(5) What is the domain of the variable or variables in this problem? Is it restricted? If so, why is it restricted?

(6) Can I estimate the answer? Check your estimate in the problem; it might give you a clue toward the solution of a problem. If you cannot estimate the answer, take a "wild" guess, and try to check that guess. It is very likely that you might pick up a clue toward solving the problem by doing this.

If you have done everything indicated here in this list faithfully and you still do not have a translation, then seek help. At least you know a great deal about the problem, and your efforts neatly written up can serve as a basis for discussion of the problem.

Remember -- realistic problem situations do not present problems in neat packages that are all of one type. You should not expect to solve one problem and then do the next thirty problems in exactly the same way. You should, however, expect to use information, concepts, and procedures developed in solving other problems to help analyze new problem situations, but problem analysis is not exciting or profitable if it involves only repetition of rote, mechanical procedures.

3.5 Exercises:

Analyze the following problems. Any statement you make should be clear, complete, and concise. Consider the following problem:

"The length of a rectangle is 3 feet less than twice its width and the perimeter of the rectangle is 48 feet. Find the length and width of the rectangle."

1. What do you know about the perimeter of the rectangle?
2. How would you compare the length and width of the rectangle?
3. What are you trying to find?
4. How can you represent the width of the rectangle?
5. How can you represent twice the width of the rectangle?
6. How can you represent the length of the rectangle?
7. Draw a figure which represents the rectangle and label the length and width.
8. Is there a basic relationship between the length, width, and perimeter of the rectangle? If so, state it.

9. Write a sentence representing the relationship between the unknowns and the given information.

Consider the following problem:

> "Oliver Baconfat and his friend Shadow Jones race snails. The rate of speed of Oliver's snail is 8 miles per week less than that of Shadow's snail. If Oliver's snail requires 5 weeks to go the same distance that Shadow's snail goes in $3\frac{2}{3}$ weeks, how fast does each snail travel in miles per week?"

(Be alert - This is a slightly different approach to problem analysis.)

10. Guess a value for the speed of Shadow's snail. (Your guess has to be > 8. Why?

11. How fast does Oliver's snail travel if you use your guess for the speed of Shadow's snail?

12. How far does Shadow's snail travel in $3\frac{2}{3}$ weeks? (Remember, $d = rt$. Use your guess for the speed.)

13. How far does Oliver's snail travel in 5 weeks? (Continue to use your guess.)

14. According to the problem both snails travel the same distance. Are your answers for exercises 12 and 13 the same? Do you see a plan for finding a correct "guess" which will make them the same? If not, try another guess and see what happens.

Now analyze the following problem in a similar manner, and write an open sentence representing the translation, if possible.

> "The flower border of uniform width around the outside of the student quadrangle at Shady Hills School has the same area as the quadrangle. The width of the quadrangle is 60 feet and the length is 90 feet. Find the width of the flower border." (Count the sidewalks cutting through the flower beds as part of the flower beds.)

15. Draw a careful sketch of the quadrangle and its surrounding flower border. Label the parts which are given.

16. What are you trying to find? Use a variable to represent it, and label this unknown part on your sketch.

17. Using the given width of the quadrangle and your variable, can you write an open phrase representing the outside width of the flower border? Label this on your sketch.

18. Using the given length of the rectangle and your variable, can you represent the outside length of the flower border? Label this on your sketch.

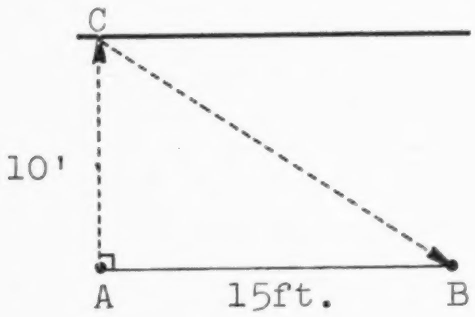
19. State a basic relationship, which you know from previous work, between the length, width, and area of a rectangle.

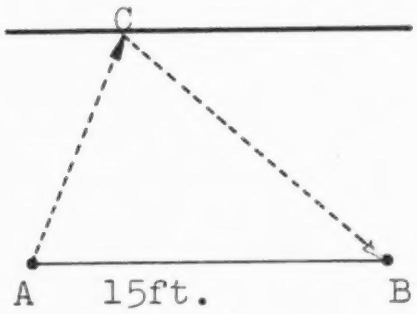
3.6 Organizing Information with Drawings or Diagrams:

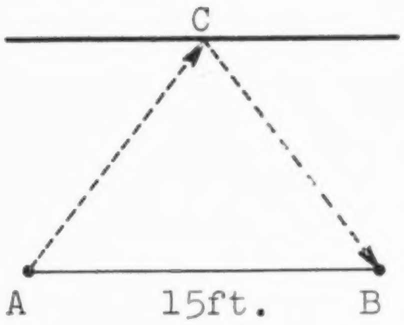
The following examples and problems are illustrations of situations where a drawing or sketch is a particularly helpful way of organizing an analysis of a problem. It is not possible to give a complete list of helpful diagrams, because each problem will have probably a unique sketch which will be most helpful in analyzing the problem. We will discuss a variety of situations in hopes that your skill in using this technique can be strengthened.

Example A: (Classroom Discussion)
Consider the following problem:
"Given two different points A and B, each ten feet from a wall, and 15 feet apart. Start from point A, touch the wall and stop at point B. Where would you touch the wall in order to arrive at B and to have walked the shortest distance possible?"

(1) Draw several diagrams (to scale or on graph paper, if possible), and experiment with different possibilities. (At least 5 or 6)







(2) Is there a geometric relationship which will help you analyze all cases? Can you draw auxilliary line segments so that there are right triangles involved in all diagrams?

(3) Can you assign variables to represent the lengths of AC, BC?

(4) Did drawing several diagrams and experimenting lead you to choose one situation as the most likely?

(5) Can you use the Pythagorean Theorem or the Triangle Inequality Theorem to analyze your conjecture?

Example B: (Classroom Discussion)
"Suppose that the sides of an equilateral triangle are doubled. What effect will this have on the area of the larger triangle compared to the original triangle?"

Make a guess now before you start!

(1) Construct two equilateral triangles as described above.

(2) Find the midpoints of each side of the larger triangle and connect these with segments.

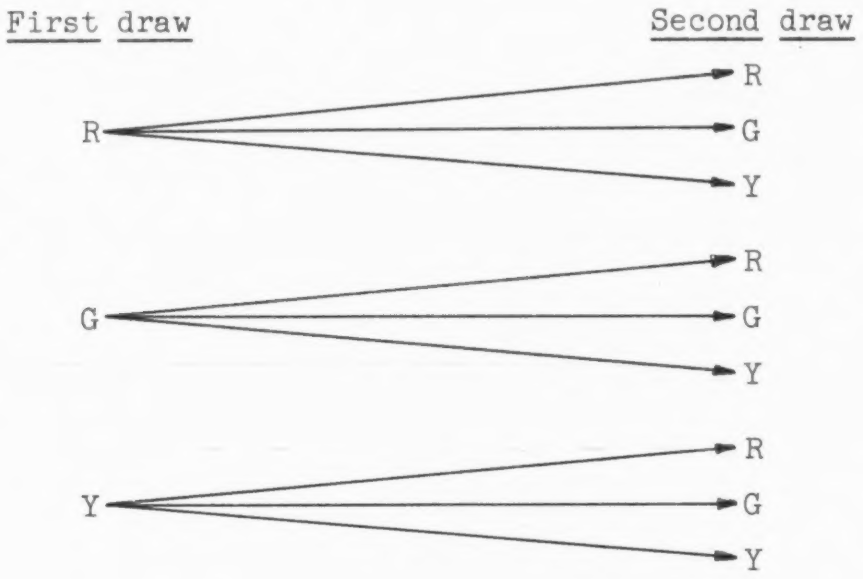
(3) Now what is your guess about the ratio of these two areas?

(4) Can you justify your guess with a proof using SSS, SAS, or the formula for the area of a triangle?

Example C: (Classroom Discussion)

"Suppose that we have a box containing 3 marbles: 1 red, 1 green, and 1 yellow. If one marble is picked at random, there are 3 possibilities. We shall call them R, G, and Y, for red, green, and yellow, respectively."

If the marble is returned to the box, and again a marble is selected at random, we have 3 possibilities for the second draw, also. The outcomes of the succession of 2 draws can be described in terms of "color on first draw and color on second draw". They are shown in this tree diagram:

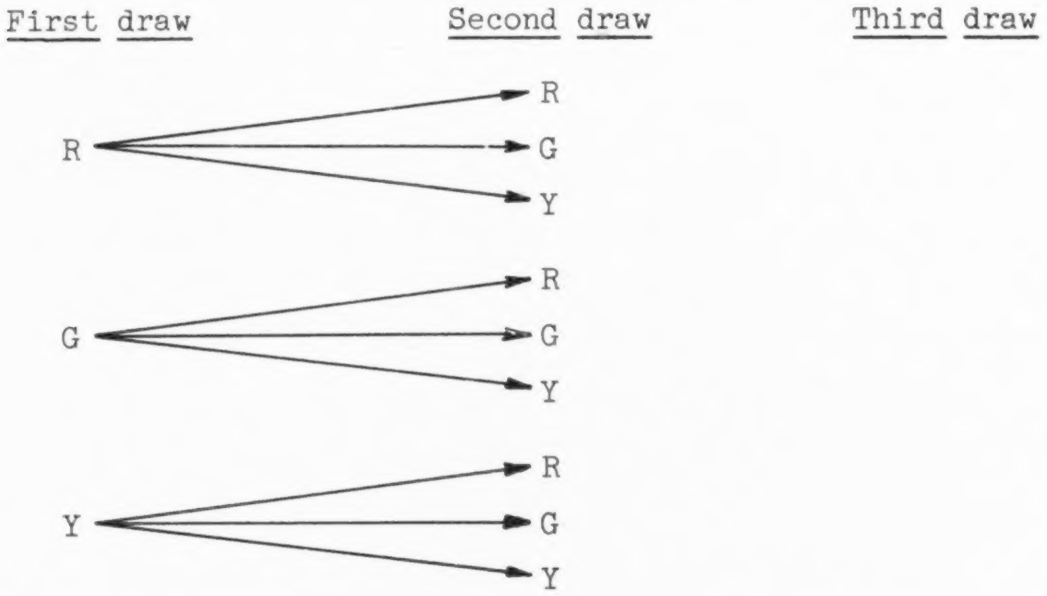


(1) On the first draw there are ______ possible outcomes.

(2) For each possibility on the first draw there are ______ possibilities on the second draw.

(3) The total number of possible outcomes on the 2 draws is ______.

Complete the tree diagram for 3 draws of the marbles assuming that each marble picked is returned to the box before the next draw.



Use the tree diagram for picking a marble 3 times to help you answer the following:

(1) On 3 draws, how many possible outcomes are there? ______.

(2) In how many outcomes is the red marble picked exactly twice? ______.

(3) In how many outcomes is the green marble picked at least twice? ______.

Example D:
A highway patrol car traveling at 100 mph starts after some bank robbers, who are traveling 95 mph on a freeway, 15 minutes after they have passed the patrol station. How long after they start chasing the robbers will they catch them?

(1) Draw two different horizontal segments starting from different points on a third vertical segment and extending in the same direction.

(2) If the length of each of these segments represents the distance traveled by the robbers and the highway patrol respectively, how do these lengths compare?

3.7 Exercises:

In each of the following problems present the given information in the form of a picture or a sketch. It is not necessary to solve the problem unless you wish to do so. However you should state your guess as to the answer based on your drawings.

1. Two students run at 15 feet/second from a point A to a line L, and walk from the line to point B on the other side of L, at 5 feet/second. To what point on line L should the first student head in order to reach B first.

2. Due to increasing population a certain city of ancient Greece found its water supply insufficient, so that water had to be channeled in from a lake in the nearby mountains. And since, unfortunately, a large hill intervened, there was no alternative to tunneling. (See Figure 1.)

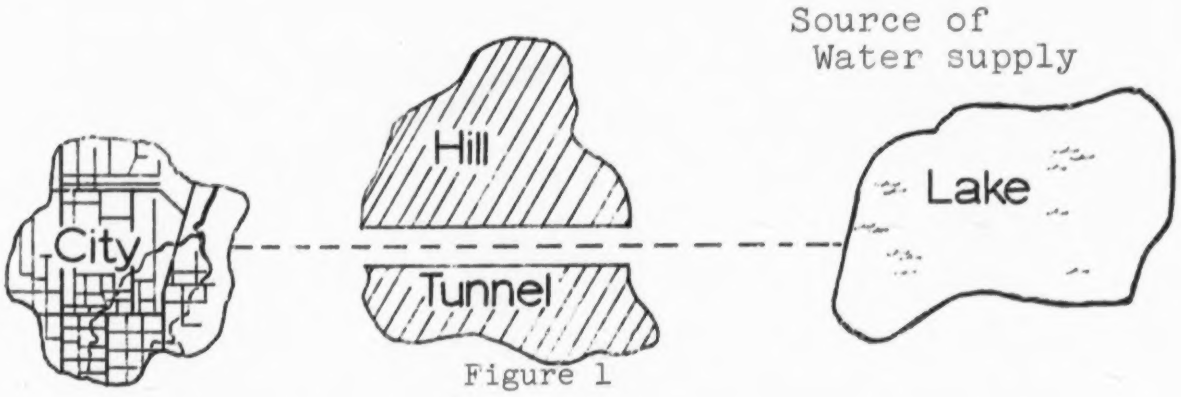


Figure 1

Working from both sides of the hill, the tunnelers met in the middle as planned.
How did the planners determine the correct direction to ensure that the two crews would meet? How would you have planned the job? Remember that the Greeks could not use

radio signal or telescope, for they had neither. Nevertheless, they devised a method and actually succeeded in making their tunnels from both sides meet somewhere inside the hill. Think about it.

3. A school is located five blocks east and six blocks north of the home of two brothers. The older brother walks four blocks east and two blocks north to his girl friend's house. They walk from there one block east to a donut shop, and then proceed directly to school. The younger brother cuts across vacant lots to a point one block directly west of school and then proceeds to school. How far does each boy walk to school?

3.8 Organizing Information in Tabular Form:

Many times it is especially interesting to arrange information in tables. In fact, such an arrangment is often times the only efficient way of gaining any insight into the solution of the problem.

Class Discussion Problems:

Example A:

There are three bus pickup points A, B, and C for taking students to school in a certain community which is considering a new school at one of two possible sites, a and b. A is four miles from a, 2 miles from b, and B is 3 miles from a, 3 miles from b, and C is 6 miles from a, 1 miles from b. 200 students are picked up at A, 250 students from B, and 225 students from C. It is desired to choose the site which will result in the minimum total time of travel to and from school by the town's student population. Which site is chosen?

(1) Can you organize a table giving the distance to a and b from each of the points A, B, and C?

(2) Can you organize a table giving the total student distances from each point A, B, and C to sites a and b?

(3) Can you now guess which site meets the required condition?

Example B:

The following list represents the results of 100 throws of a die. How does the occurrence of the results compare with the theoretical probability of rolling a 1, 2, 3, 4, 5, or 6?

100 throws of a die

43553	53344	14166	53213	46451
54563	41353	35335	65536	64112
43253	62454	53263	33423	21531
24131	64235	26563	22522	21355

Example C:

Is there a relationship between the length and width of the leaves of a particular tree?

(1) Select 20 leaves from a tree or bush and measure the length and largest width of each leaf.

(2) Construct a table listing this data and include the sum, the difference (length - width), the product, and the ratio (length - width).

(3) Can you now state whether a relationship exists for your leaves? If there is one, which method of comparison gave you this information?

3.9 Exercises:

In each of the following problems, present the given information in tabular form and try to answer the questions about this data.

3.10 Estimation Process

In this section we wish to focus your attention upon the technique of "guessing" an answer and the information which can be derived from this approach to problem analysis.

Classroom Discussion:

Example A:
Suppose that we sketch a wire around the earth at the equator. (Assume that the earth is a smooth sphere of diameter 8000 miles.) If we cut the wire, insert a piece one foot long and then hold the wire above the surface so that it is the same distance above the earth all the way around, how far above the surface will the wire be? (Use $\pi \approx \frac{22}{7}$.)

(1) Let's guess an answer and see if it is too large or to small. First guess: ______ feet.

(2) The diameter of the new circle of wire would now be 8000 miles + 2 (? feet). Why 2 times your first guess? (Don't forget to change 8000 miles to feet.)

(3) How does the circumference of the new wire circle compare to the original wire? (How are you going to compare these two numbers; by addition, subtraction, division, or multiplication?)

(4) Was your guess too large or too small?

(5) Unless you see how to work the problem directly, revise your guess and check your results again.

Example B:
A farmer found that it took 240 feet of fence to go around his rectangular farmyard. He noticed that one of the sides was 40 feet long. How long are the other sides?

(1) Let's guess ______ feet for the width.

(2) How do you check your guess? Was it too large or too small?

(3) Can you solve this problem using a sentence?

Example C:

The radar operator on an aircraft carrier detects a contact moving directly toward the carrier. He estimates the distance to the contact at 400 miles and the speed of the contact at 350 miles per hour. How long will it take one of the carrier's planes to intercept the contact if it flies directly toward the contact at 450 miles per hour?

(1) Guess the time it takes the carrier to intercept the contact. ______ hours.

(2) How do you check your guess?

(3) How far does the contact fly in the time you guessed?

(4) How far does the aircraft carrier plane fly in the time you guessed?

(5) Can you use the method you used in checking your guess to solve the problem?

3.11 Exercises

In the following problems guess the answers to the questions and try to see if the method you used to check your guess can be modified to solve the problem.

3.12 Problem Analysis Based on Analogy:

Sometimes problems in their original form are too complicated to solve. These problems can often be analyzed by simplifying the situation and then looking at the simpler model. We would like to look carefully at this method on the following problems.

Class Discussion:

Example A:
What is the longest line segment that can be drawn in the interior of a sphere from a given point on the sphere to a different point on the sphere.

(1) Suppose we simplify our situation and consider a circle and try to figure out what the longest segment would be from a given point on a circle.

(2) If we draw a segment from the given point through the center of the circle, how does this segment compare with every other segment in the circle from the given point?

(3) Can we extend this analysis to a sphere.

Example B:
Mr. X has to go to a town T, 78 miles from his house. He can take a bus at 11:50 AM and, 35 minutes later the train, which gets him to his destination in 45 minutes.

If he decides to drive, and he can count on an average of 50 mph, when would he have to leave to get to T at the same time?

Simpler Form:

Part 1: How long does the trip take by bus and train leaving at 11:50? What is the time of arrival?

Part 2: In what time can a distance of 78 miles be covered, traveling at a rate of 50 mph?

Some additional suggestions:
Some problems in which the student is faced with - the problem of constructing geometric figures with indufficient data, or that there is not a unique solution.

GRADE 8 - CHAPTER 4

APPENDIX

THE USE OF FUNCTIONS IN PROBLEM SOLVING

In accord with our increased emphasis on function we point out here how to use functions in solving some of the problems mentioned in Chapter 4, Problem Solving. In addition we also discuss the isoperimetric problem of determining the rectangle of fixed perimeter with maximal area. It turns out that this provides a classic example of the interplay between synthetic geometry and analysis.

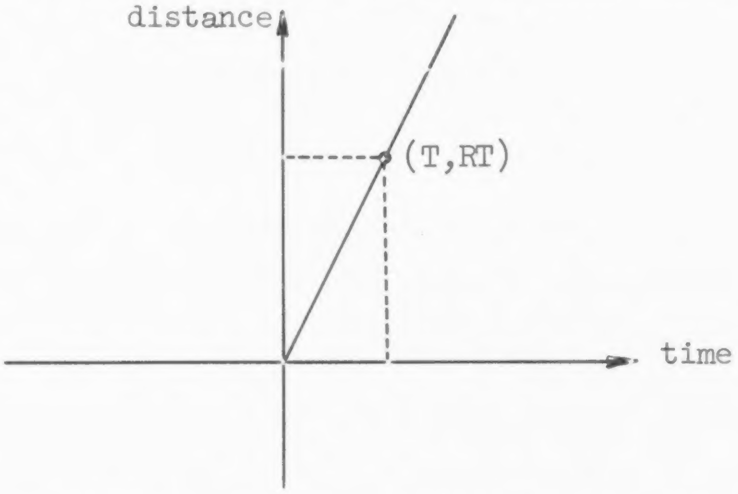
Of course we realize that not all problems are facilitated through the use of functions. However, if the student, when stumped, will ask "What function lurks in the background" he may find that he has a new outlook on the problem and a new tool which will prove effective. Moreover, since graphing of functions creates a model of the problem, this attack will show, particularly for simple problems, the unity of a variety of problems. Such an example is provided in these pages in a consideration of Problems 8, 17, and 27 of Chapter 4, done here on pages 6-7. It should be clear that this general method will apply to any rate problem, including the well known work and mixture problems. Indeed, to make our point we might claim that by an introduction of a function, the student has a tool of such wide applicability that this gives him a general method for solving all the traditional problems of high school! (We grant the existence of counterexamples!)

As a first example of the sort of thing that we can do, although we would not recommend this as the first example for the student to see, we shall analyze the "Two Satellite" problem on page 3, Section 3.3.

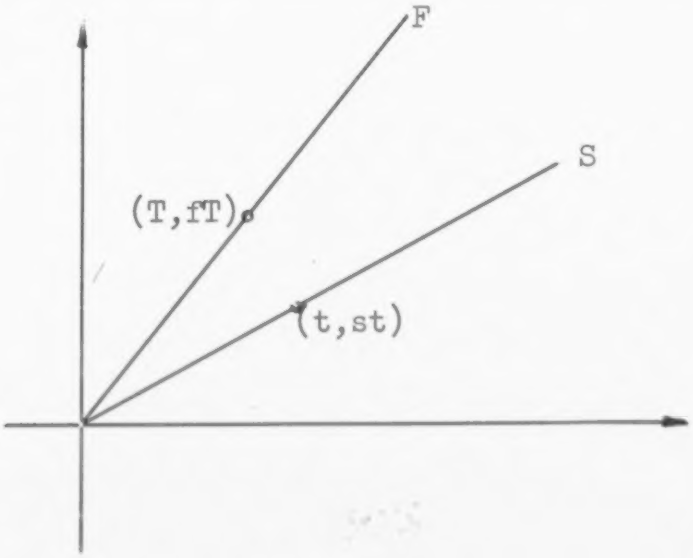
Two satellites are placed in the same sized circular orbit. The first satellite is traveling 8 miles per minute faster than the second. The faster satellite requires 2 minutes less time for the 30,000 mile trip around the earth that the slower satellite. Find the rate of speed of each.

Analysis:

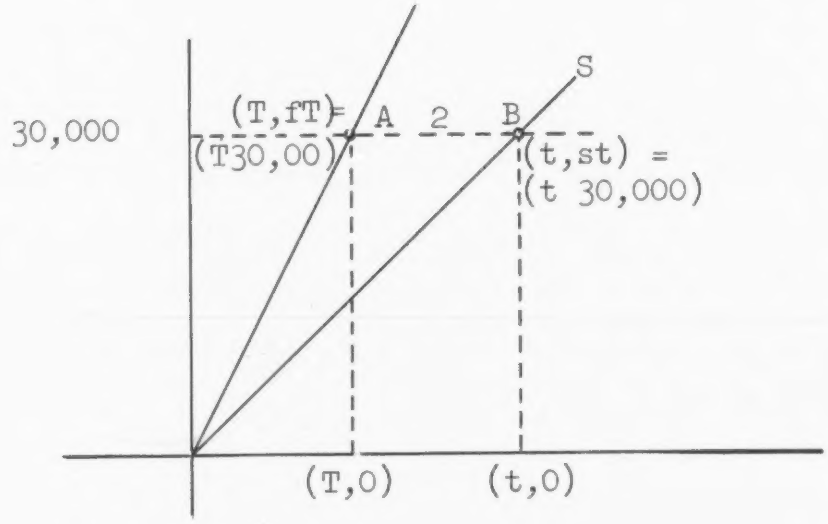
The student must see that the basic functional relationship here is Distance = Rate × Time. That is, that the function $D : T \rightarrow RT$ associates the time traveled with the distance traveled, which is computed by multiplying the rate of travelling by the time. This multiplication, (RT), has been graphed earlier and looks like this:



In this problem we have two satellites, a fast one, F, and a slow one, S. Let us graph the appropriate function of each. We can't do it specifically because we don't know, yet, their respective rates. But we do know that the rate of F, call it f, is greater than the rate of S, call it s. That is $f > s$. And from past experience with multiplication functions we know that their graphs should be related thusly:



Now each satellite goes 30,000 miles. Enter on graph:



We also know that the difference in time for each to go 30,000 miles is 2 minutes. Thus the difference between the first coordinates of B and A is 2, that is $t - T = 2$. Thus

$$30{,}000 = f\ T = s\ t = s(T + 2).$$

We also know that f is 8 greater than s, so $s = f - 8$. Thus

$$30{,}000 = f\ T$$
$$30{,}000 = (f - 8)(T + 2).$$

Now we can use the first equation to relate f and T, $f = \frac{30{,}000}{T}$, thus

$$30{,}000 = (\frac{30{,}000}{T} - 8)(T + 2)\ .$$

It is interesting to contrast this solution with the "Box" solution. We reproduce this solution in its entirety; it is essentially self explanatory.

	R	× T	= D
F	f	$\frac{30,000}{f}$	30,000
S	$f - 8$	$\frac{30,000}{f - 8}$	30,000

Now S's time - F's time = 2. Thus $\frac{30,000}{f - 8} - \frac{30,000}{f} = 2$.

Note: This equation is even easier to solve that the one we derived. The student who can solve the problem is this way has no need of analysis procedures.

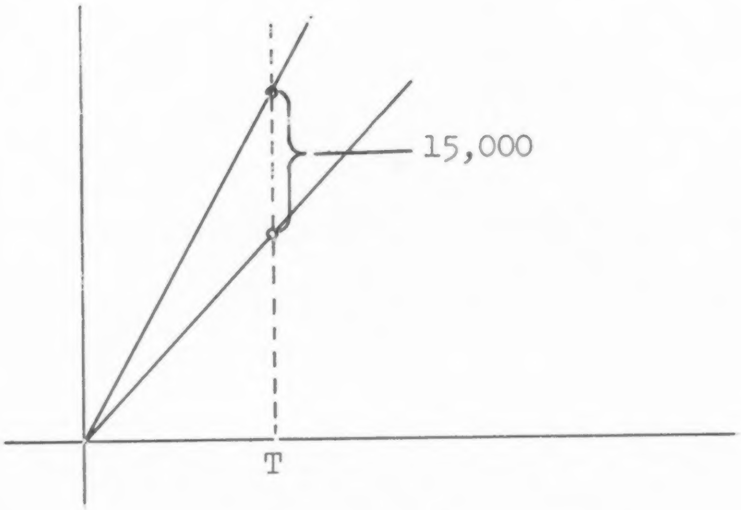
We do not give up the function approach easily. As we said, the function approach is for the student who has tried everything -- and has not been successful. It is fair to point out that for the usual high school problem, the solution itself is of little importance and has no value in the market place. However, the method of solution may indeed have a value in the market place -- certainly the ability to give new ones with new insights is a highly prized one.

The function approach gives promise of being applicable to less stereotyped kinds of problems. The approach does have some of the aspects of generality; in particular a variety of information falls out of the general solution. For example, our same graphs give us solutions for the next two problems:

SECOND two satellite problem:

If F and S are fired into orbit from the same place and time, when will they be on opposite sides of the earth?

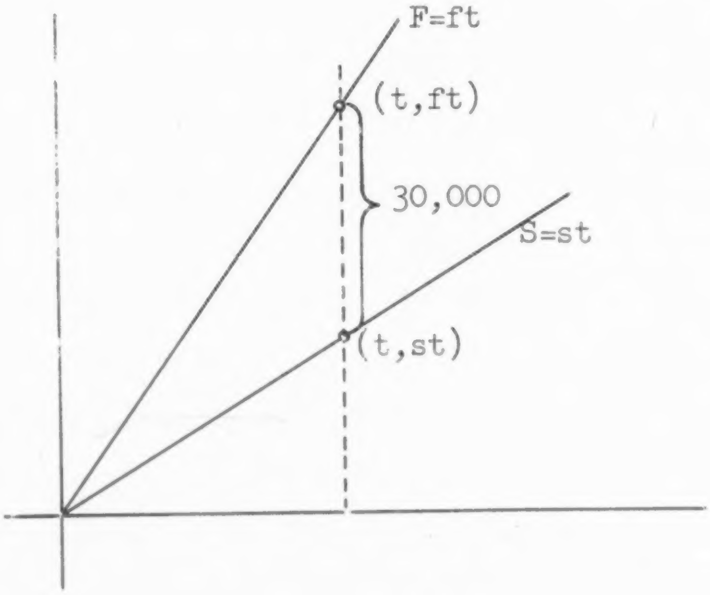
The graphical interpretation is essentially the same except that we look for points with the same time coordinate which are 15,000 units apart on the distance scale.



(We have ignored here the point that we must first find the rates.)

THIRD two satellite problem:

Two satellites, F and S are fired into the same orbit which is 30,000 miles in circumference. F is travelling 8 miles per minute faster than S. When will the faster one rendezvous with the second?



We take our analysis from the graph,

$$ft - st = 30{,}000$$
$$(f - s)t = 30{,}000$$
$$f - s = 8$$
$$t = \frac{30{,}000}{8}$$

As to the many excellent points which Chapter 4 makes we add that many can be rephrased in terms of functions. Some of them should be! We shall now cite section and page numbers of situations in which some additional clarity can be gained by an alternate statement using function. We put in quotation marks an alternate way of expressing his question or answer.

Page 2. Section 1.2. "Consider the function $(R,W) \to R - \frac{W}{4}$. Try to write an English sentence which clearly interprets what is meant."

Page 3. "Write an English phrase which describes the association mentioned in the functions below".

1. $t \to 60t$
2. $(L,W) \to 2L + 2W$ .... etc.

Page 4ff. Section 1.8. "In each of the exercises determine a function which interprets the English phrase ....."

1. $t \to 3t$
2. $(k,n) \to k + 3n$
6. $x \to x + (x + 1)$
19. $(x,y) \to \max(x,y) + x - y$

    or

    $$(x,y) \quad \begin{cases} 0 & \text{if } x > y \\ y(y - x) & \text{if } x \leq y \end{cases}$$
29. $j \to j + 500$
30. $j \to j - 500$
31. $(j,t) \to 6(j + 500)t$
32. $(j,t) \to 6(j - 500)t$

Page 7. Point out that an equation such as $35x + 70 \cdot 40 = 50(x + 40)$ is the statement that the graphs of two functions f and g:

$$f : x \to 35x + 70 \cdot 40 \qquad g : x \to 50(x + 40)$$

intersect. (It is of course conceivable that they won't intersect.)

Page 9. Satellite Plating Problem (A rate problem). Analysis:
Key question: If x ounces of gold are used, what is the cost?
The function is

$$x \to 70 \cdot 40 + 35x \,.$$

If the student does not see the function right off the question can be asked in terms of 0 ounces of gold, 1 ounce of gold, 10 ounces of gold, etc.

Please note that this approach is not a "guess the answer" approach. It is rather "What's going on" approach. Indeed I feel that pressing the student for a complete answer to the whole problem is wrong and tends to make him gun shy. George Polya not withstanding!

Page 10: Plot various graphs of functions on either side of (=). This will be tough for several variables situation; but in the two variables case, letting one act as a parameter and plotting for different values of the parameter may give real insight.

Page 11: The sail problem. We can write a function, $a \rightarrow \frac{a(a - 6)}{2}$. However this is an instance of forcing the use of function. Indeed, I find it hard to improve on Chapter 4's solution. Can you?

Page 12ff: Problems 1-14 seem routine. Indeed, on most "age" and "number-digit" problems I have found the introduction of function to be uncomfortable pedantry. However it may be that some of these examples, especially rate problems must be done with function to pave the way for greater things.

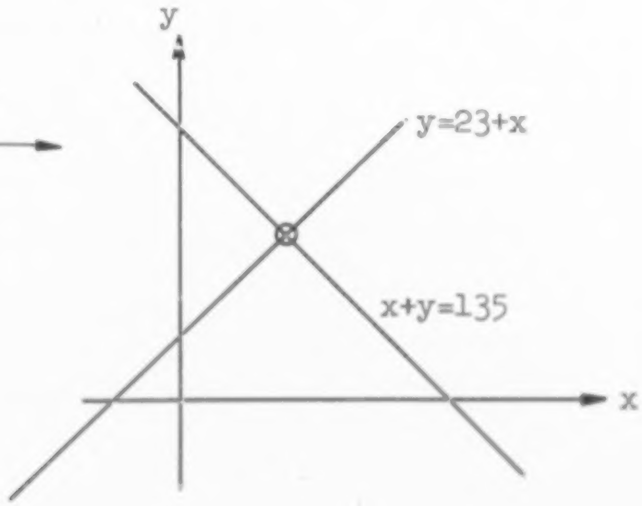
Problem 8 deserves a little more attention:

If x and y are the two numbers we have

$$x + y = 135 \qquad \text{and}$$

$$y = 23 + 3x$$

Graph each: SO!



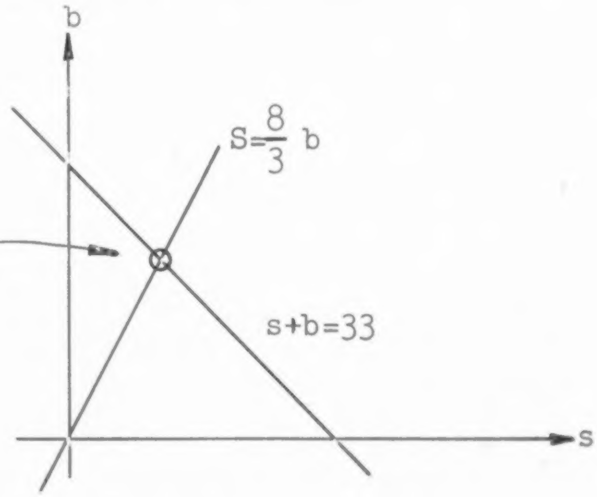
Problem 15: %: $n \rightarrow \frac{n}{100}$ tells us that $500 \rightarrow \frac{500}{100} = 5$. Thus from the function $x \rightarrow x + 5x$ we obtain the rule for the total take from selling x books.

Problem 17: s is snarfs, b is bminfs:

$$s + b = 33$$

$$s = \frac{8}{3} \cdot b$$

Graph each: SO!



Recall Problem 8

Problem 22: $x \rightarrow 2x + 2$ tells us that if x is the amount of money Ollie has now, the shoes cost $2x + 2$. Solve $2x + 2 = 20 \cdot 50$.

Problem 27: If a is the amount of the secret ingredient and h is the amount of hydrogen then

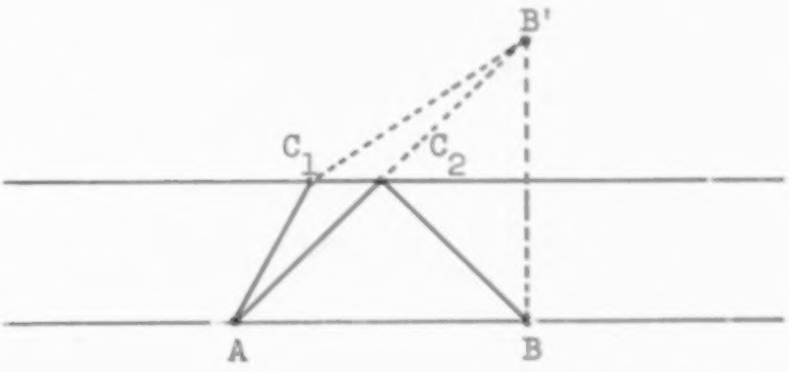
$$s + h = 4$$

$$h = \left(\frac{17}{100}\right)s$$

Graph: Note similarity with Problems 8 and 17.

*** Problem 26: If w is the number of papers sold on weekdays and s is the number of papers sold on Sunday we study the function $(w,s) \rightarrow w + 2s$. The side conditions $w + s = 1700$ and $w + 2s = 2200$ set up the problem. In connection with this problem Jean Calloway has pointed out that to motivate the need (usefulness) of the function, this problem should be expanded by giving a table of sales and profits for several weeks. There is nothing like repeated pencil pushing to motivate the need for a single function to do it all at once! This excellent suggestion for motivating the need for a function should be applied in many places.

Page 21: (I can't pass up this remark.) In Example A, try reflecting the point B across the line on which C lies!



Page 23: Example D: See the "Two Satellite" problem, second part!

Page 24: 1: Function approach clearly works. Try any one!
2: I don't see any function lurking here!

Section 3.8 Tables are functions!

ISOPERIMETRIC PROBLEMS

This class of problems provides one of the nicest interplays of synthetic geometry and analysis in the entire industry! We probably cannot hope to show, analytically, that among all geometric figures with a fixed perimeter the circle has the largest area. However, as Pamela Ames has pointed out, cut and paste techniques can lead to this conclusion.

A more restrictive problem: Among rectangles with a fixed perimeter, which ones have the largest area ? can be solved both by cutting and pasting and, more precisely, by analysis. The actual amount of precision depends of course on the analytic tools at our disposal.

Cut and Paste Solution:

Give each student a dozen straws (soda type) and some grid paper.

Directions:

1. (a) Take one straw and cut it into four segments that can be used to form a rectangle. Note that the method of cutting will take some good thinking (or even help) the first time. Hopefully, they will eventually cut the straw into two segments and then cut each segment into two congruent segments.
   (b) Place the rectangle formed by these segments on the grid paper and trace the region.
   (c) Count the squares to get a measure for the region (the area).
2. Repeat the above procedure with each of the other straws using segments of different lengths.

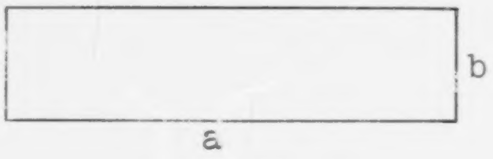
AHA: The maximum rectangular area with a given perimeter is a square!

Analytic Solution -- beginning with a story problem:

> John has 24 feet of fencing to make a rectangular shaped pen for his dog. What should its dimensions be so that the dog will have the most play area?

(If the student knows that a square has the largest area amoung all rectangle of fixed perimeter, then the solution is trivial: $4s = 24$, $s = 6$.)

We first draw a schematic picture of a rectangle:



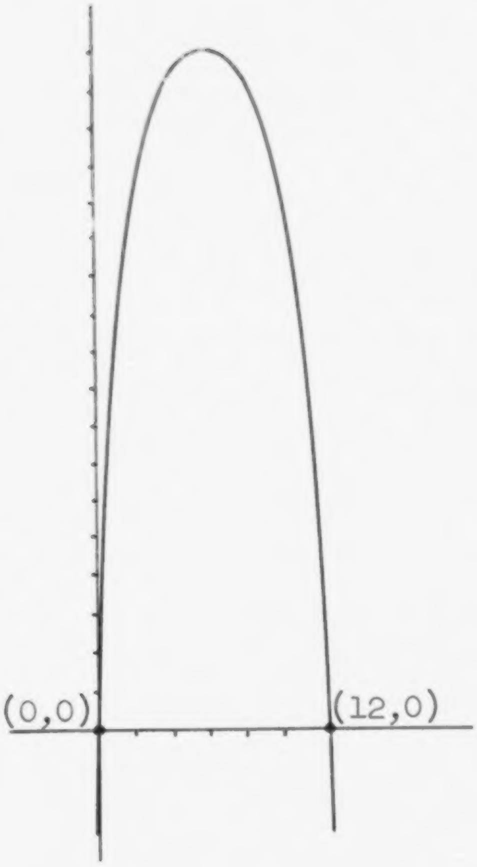
$$\text{Perimeter} = 24 = 2(a + b)$$
$$\text{Area} = ab$$

Problem: How to choose a and b so that ab is maximised subject to the side condition $a + b = 12$? The perimeter relation yields $b = 12 - a$ and so we seek the maximum of the area function

$$a \rightarrow a(12 - a) = -a^2 + 12a$$

Or: Maximise $a(12 - a)$.
Plot the graph of this function:

If graphing is the only tool at the disposal of the student, then the maximum $(6,36)$ must be read from the graph. From the analysis he then concludes it is a square.



If completion of the square is a technique at our disposal we argue:

$$-a^2 + 12a = -(a^2 - 12a + \_\_) + \_\_\_ = -(a^2 - 12a + 36) + 36$$
$$= -(a - 6)^2 + 36.$$

Now the expression $36 - (a - 6)^2$ clearly has a maximum when the subtracted term (which is positive) is at a minimum, and since the subtracted term is a perfect square this happens when $a - 6 = 0$, when $a = 6$. From this $b = 6$ and the max area is 36.

Another type of solution, variation of a parameter, is also available to us in an intuitive and graphing form.

Consider $ab = K$. For different values of K, what is the graph?

Try, in turn, $K = 1, 2, 4, 8, 16, 32, 36, 40, 50, 100$.

Note that each curve of this family is symmetric about the line $a = b$. (See Next Page.)

After plotting these curves on the same graph, let us now plot the condition

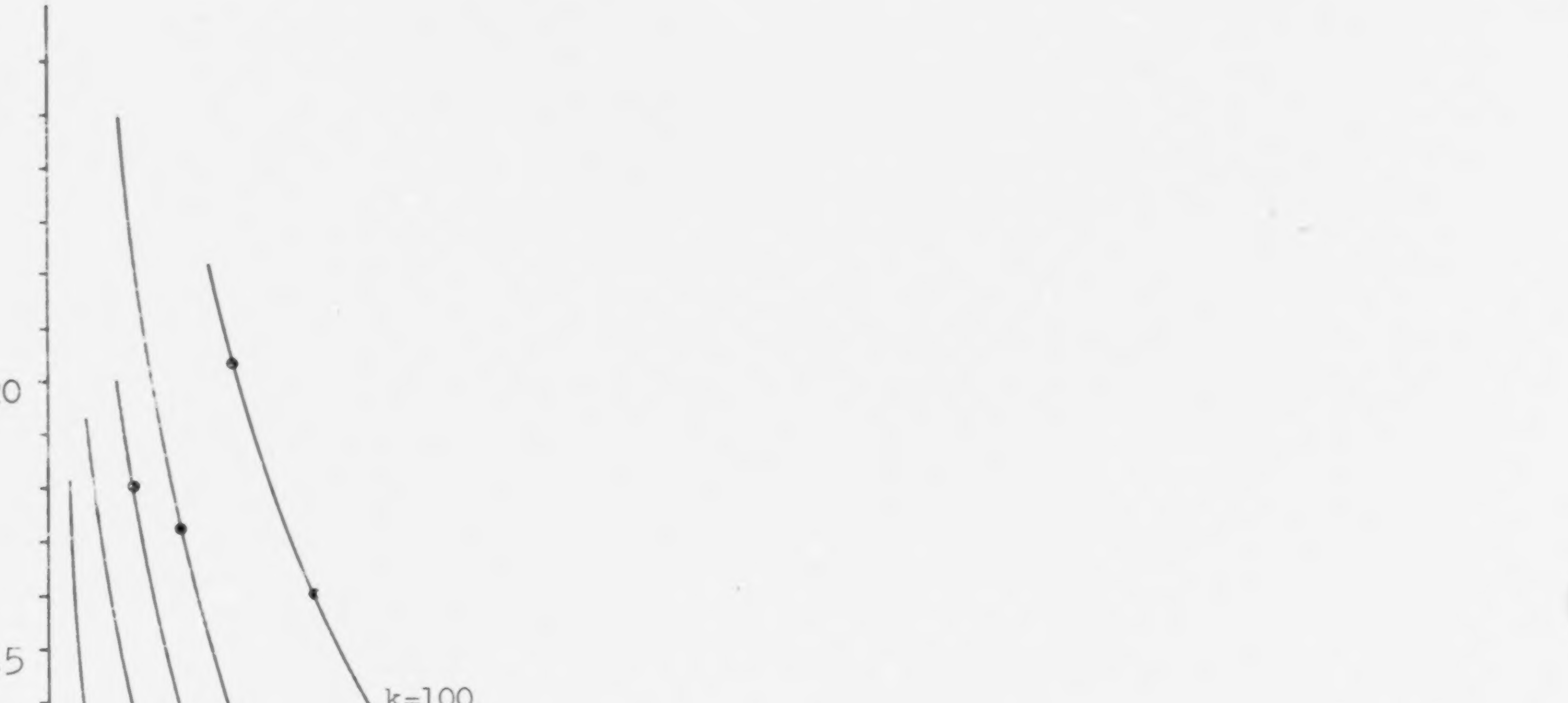
$$a = 12 - b.$$

This line cuts some of the curves $ab = K$, but not others. Since we seek a point of intersection of $ab = K$ and $a + b = 12$ we must select a value of K for which the curve ab = K and a + b = 12 intersect. However, we want K to be as large as possible. Intuitively this will be the value for which the line a + b = 12 is tangent to ab = K. From the symmetry of the figure the point of tangency will occur when a = b and thus a = b = 6, i.e., at the point of intersection of a + b = 12 and a = b -- This conclusion is not one we could have easily justified without the variation of parameter method.

Or, pursuing this more analytically, for any K, if ab = K and b = 12 - a then

$$a(12 - a) = K \quad ; \text{ or}$$
$$0 = a^2 - 12a + K$$

which will have one solution for a if and only if the discriminant $12^2 - 4K = 0$, that is when K = 36, and thus a = 6 = b.


20
15
12
10
8
5
4
2
1
k=100
k=50
k=36
k=32
k=16
k=8
k=4
k=1
1
2
4
5
8
10
12
16

A variation of this problem is to assume that one side of the rectangle is given free along one line; the other three sides to have a fixed sum. The equivalent of the story problem is to imagine that John is to build his pen along the back wall of his lot, the back wall being 60 feet long. He still has 24 feet of fencing. What shape (rectangular) should it be? This is a particularly nice problem since a square is not the answer, and the student's intuition may not be as accurate. The analyses of each variety are essentially the same.

An important variation of greater difficulty is to remove the restriction of rectangular regions -- try a triangle, and various other polygons. What happens as the number of sides increases? Try for the circle as the solution to the general isoperimetric problem in the plane.

At some point in this experimentation the students may want to replace straws with string. One nice observation will be that whatever figure is considered, making it convex (if it is not already) improves the size of the area at no cost of perimeter.

GRADE 8 - CHAPTER 5

NUMBER THEORY

Background Assumptions:

A good bit of experience with the properties of the arithmetic operations as applied to integers. Also a good deal of familiarity with integers -- multiplication facts, factoring, dividing large numbers, etc.

Purposes:

1. To teach the meaning of the unique factorization theorem for integers and its use in computation; tests for divisibility and tests for primality.
2. Study of proof in some easy situations where short arguments suffice. Discussion of "If ..., then..." statements, converse, negation.
3. To pursue some ideas of mathematical interest just because they are interesting -- mathematics is more than just models of the real world.

5.1 Even and odd integers.

Motivation

Sometimes we are interested in only some aspects of a number. Street addresses - even numbers on one side, odd numbers on the other. Square dances - squares of 4 couples. Bridge or other partnership card games. Circuits, computer storage - only interested in one of two states.

The "facts" about even and odd numbers have probably already been discussed in Grades 4 - 6. Review these by asking questions like:

1. Is the sum of two even numbers even?
2. Is the sum of an even number and an odd number even?
3. Is the sum of two odd numbers even?
4. Is the product of two odd numbers odd?
5. Is the product of two even numbers even?
6. Is the product of an even number and an odd number even?
7. If the sum of two numbers is even and one of them is odd, what can you say about the other one?
8. If the product of two numbers is even and one of them is even, what can you say about the other one?

Collect all of this information in the form of addition and multiplication tables for even and odd.

After the children have worked with enough examples, try to get them to give a definition of "even". Of "odd". Let this discussion lead into rather more formal statements of the theorems about even and odd numbers.

5.2 Informal discussion of statements and proof.

Some discussion about mathematical statements (true or false), about the form they usually take ("If ..., then..."), and quantifiers ("some", "all", "There is"). Try to reformulate the statements about even and odd as theorems in the "If ..., then...." form. Ask questions about the converse, etc.

For example, if the students were led to propose as a definition of "even": n is even if it has a factor 2; or n is even if there is an integer n' such that $n = 2n'$; then a direct proof using the distributive property can be given of the theorem:

"If m and n are even, then $m + n$ is even."

Point out that if m and n are not both even, then the statement makes no claim about $m + n$.

Ask about the truth of:

"If $m + n$ is even, then m and n are even."

After you have a definition for even, try to get one for odd. If odd turns out to be "not even", then try to say what that would mean; i.e., n is odd if and only if there is an n' such that $n = 2n' + 1$.

This should give a natural lead-in to the general definition of divisibility and the division algorithm. As an exercise it might be suggested that the student make a flow chart for the division algorithm.

Some theorems which can be given for which the student can give a more or less formal proof at this stage are:

1. If m and n are even, $m + n$ is even.
2. If m and n are even, then mn is even.
3. If m is even and $m + n$ is even, then n is even.
4. If m and n are odd, then $m + n$ is even.
5. If m and n are odd, then $m - n$ is even.
6. If m and n are odd, then mn is odd.
7. If m is even, m^2 is even.
8. If m is odd, then m^2 is odd.

To show that a proposed statement is false; e.g., "If mn is even, then m and n are even", point out that it suffices to find one example in which the theorem is false, $5 \cdot 6$ is even but 5 is odd. This might lead to a general examination of the unmentioned quantifiers in the statements made above.

(While negations of "if..., then..." statements as well as contrapositives might naturally arise in this discussion, postpone considering such things until later as there are too many complications for a first try at this sort of thing.)

5.3 Factors, divisibility, tests for divisibility and the division algorithm.

Motivation - See Section 10-1 in First Course in Algebra (pp. 248-51)

With very little more effort, one can get a definition of divisibility by 2, 4, 5, 10 and also 3, 6 and 9. While devising these tests one could develop the division algorithm: $a = bq + r$ where $0 \leq r < b$.

In the exercises for this section one could extend the proofs of the previous section to prove theorems such as:

> If d divides a and d divides b, then b divides a + b; a - b; ka; kb; ab; and ax + by for any integers x and y.

As a lead in, theorems such as:

> If 4 divides n, then 2 divides n.
> Ask about the converse.

The following might be proposed:

> If 2 divides n and 3 divides n, then 6 divides n. Converse.

One might also propose the theorem:

> If d divides ab, then d divides a or d divides b.

Let the students discover that this is true if d is prime, but not necessarily true otherwise.

5.4 Prime Numbers, the Sieve of Eratosthenes, Prime Factorization.

There are versions of this material in Chapter 10-2 and 10-3 of First Course in Algebra and Chapter 11-2 in Introduction to Algebra (pp. 464-476). However, the sieve can be done in such a way that other questions arise. The conclusion that if no prime less than or = $\sqrt{\pi}$ divides n, then n is prime can be obtained.

You may want to raise the question: "How many primes are there?" If the division algorithm has been discussed in 5.3, Euclid's proof that there are infinitely many primes, can be turned into a constructive proof showing that given any set of primes there is always another prime not in the given set. (In the Teacher's Commentary the analogous argument for primes of the form $4k - 1$ can be given.)

Following this section something similar to the last two sections of Chapter 10, FCA, pp. 266-282 might be added, since the laws of exponents have not appeared so far in the outlines for Grades 7 and 8.

5.5 The Euclidean Algorithm and the GCD.

Purpose:

As a follow-up to the message in Chapter 4 on Problem Analysis, and as a review and different way of looking at the work of Chapter 2, Grade 7, on graphing of straight lines, one can pose the question of solving equations of the form $ax + by = c$, where a, b and c are integers, for integers x and y. One version of this material is contained in ESSAYS ON NUMBER THEORY II, Chapter 4, pp. 19-26.

Rationale:

This section would have to be included as an example of "interesting" mathematics. It would be hard to defend the position that this is something everyone should know. There are easier and more direct ways of getting the GCD and then the LCM for adding rational numbers! On the other hand, it certainly does show the work on linear equations in a new light and offers opportunities for arithmetic manipulations with another goal in mind.

5.5a (Alternative)

See Hassler Whitney's paper on the "Introduction of Mathematical Concepts" (p. 4) for another and shorter way to introduce the GCD. This introduces modular arithmetic and could lead to GCD and, he claims, to F.T.A.

When the Euclidean Algorithm is developed, there are two very nice flow charts which could reinforce and clarify this algorithm on p. 30 of the New Orleans Conference Report, March 14-18, 1966.

GRADE 8 - CHAPTER 6

THE REAL NUMBERS REVISITED - RADICALS

Background:

Exponents. (NOTE: While Chapter 3, Grade 7 has some work with scientific notation, rationals in expanded form with extension of exponent notation to negative exponents, there is not at present in the seventh or eighth grade outlines a specific place where the laws of exponents are restated and worked with. To assume that this was done before the seventh grade and need not be done again until the eighth grade is a mistake. It should somehow be worked into the seventh grade outline and probably reviewed and restated in the NUMBER THEORY chapter of the eighth grade after the unique factorization theorem.)

Solution set of an equation,

Order: If $0 < a < b$ and $0 < c < d$, $ac < bd$.

Geometric construction for separation of segment line congruent segments.

Absolute value: $|x|$.

Properties of even and odd numbers.

Decimals, Square Roots, the Real Number Line (Ch. 10, Grade 7)

Pythagorean Theorem

Purposes:

Review of the real number system, motivated by the consideration of certain problems which do not have rational solutions; decimal notation for rational and irrational numbers; meaning of radical and practice in computations involving radicals; functions involving radicals; review and summary of properties of the real numbers and properties of subsets of the real numbers.

6.1 Motivation

Suggest problems requiring irrational numbers in their solution such as:

1. A biologist has a cube with edge 3 meters, in which a man is to be enclosed for a specified time. This cube is just large enough to provide him with sifficient air for the time he is to occupy it. He now wishes to build a cube which will be just large enough for two men -- that is, to build a cube which has double the volume of the first. What should be the length of its edge?

   This is the same problem that Greek geometers tried to solve two thousand years ago. They set different restrictions on the problem, however; they required that the edge be determined by geometric construction, using only compass and straightedge. It has been proved that the compass-straightedge construction cannot be done. Can you solve the problem using numbers? Try different lengths for the edge, to see whether you can find an edge which gives a volume of 54.

2. Present a description of the golden section. This could be made the subject of a film strip. See the Disney film, "Donald in Mathemagic Land", part of which relates to the golden section. Also see Nicolet film number fourteen -- strophoid, golden section, and vases.

6.2 Review of Facts about the Real Number System. (See MJHS, Vol. 2, Chapter 6, pp. 235 ff.)

6.2-1 Notation for real numbers. Use exercises to recall:

(a) Every infinite decimal names a real number.

(b) If the infinite decimal is a repeating decimal, it names a rational number; if the infinite decimal does not repeat it names an irrational number.

Possibility: If repeating decimals are written in expanded form, it might not be a bad idea to spend a few minutes talking about the meaning of the symbol "..."; e.g.,

$$
\begin{aligned}
.\overline{3} &= \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \ldots \\
&= 3\left(\frac{1}{10} + \frac{1}{10^2} + \frac{1}{10^3} + \ldots\right)
\end{aligned}
$$

The notions of limit and of infinite series are both ideas which are really difficult to teach and to learn. Early exposure and a longer period of time to get used to the idea might prove helpful in later work. The student has already seen the disguised manipulation:

$$
\begin{aligned}
r &= .\overline{3} \\
10r &= 3.\overline{3} \\
9r &= 3 \\
r &= \frac{1}{3}\,.
\end{aligned}
$$

Writing out the repeating decimal in expanded form might pave the way for future belief in the validity of work with infinite series. We don't suggest doing any more than simply noting that the indicated infinite series above is another way of denoting the rational number $\frac{1}{3}$.

(c) A rational number may be named by a fraction of the form $\frac{a}{b}$, where a is an integer and b is a counting number.

(d) A numeral of the form $\sqrt{a}$ or $-\sqrt{a}$, where a is a counting number, names a real number; if $\sqrt{a}$ is the product of two equal integral factors, a is a rational number; otherwise it is an irrational number.

6.2-2 Proof that $\sqrt{2}$ is irrational. (Alt. version -- ISSM, Vol. 2, pp. 362ff.)

6.3 Roots of Numbers. (See FCA, Chapter 11, and PFCA-H, Chapter 15)

1. Square Roots

Definition of $\sqrt{a}$ and $-\sqrt{a}$ with $a \geq 0$.
Discussion of solutions of equations of the form $x^2 = a$ with appropriate restriction on a.

For all real numbers x, $\sqrt{x^2} = |x|$, $-\sqrt{x^2} = -|x|$.

In the exercises use function idea to practice finding the domain and range of functions such as:

$$f : x \to \sqrt{x^2}$$

$$g : x \to \sqrt{x - 3}$$

$$h : x \to \sqrt{x^2 - 2} \text{ , etc.}$$

2. Definition of nth root of a. Solution set of $x^3 = a$; $x^4 = a$. (MJHS, Vol. 2, pp. 272ff.)

Domain and range of functions $f : x \to \sqrt{x^3}$; $g : x \to \sqrt{x^4}$; etc.

3. Should we introduce $x^{\frac{1}{2}}$, $x^{\frac{1}{4}}$, etc., here or wait until the chapter on the exponential function?

6.4 Computations with Radicals.

1. Use of factorization theorem in finding roots:

Ex. $\sqrt{441}$, $-\sqrt{1225}$, $-\sqrt[3]{\frac{125}{64}}$, $\sqrt{11,56}$, $\sqrt{.008}$, etc.

2. Irrational square roots.

Recall of the theorem: If n is a counting number and $\sqrt{n}$ is rational, then $\sqrt{n}$ is an integer.

Proof of theorem: If $0 < a < b$, then $\sqrt{a} < \sqrt{b}$.

Review of iteration method for approximating roots.

3. Product of square roots.

   (a) Theorem: $\sqrt{a} \cdot \sqrt{b} = \sqrt{ab}$, $a \geq 0$, $b \geq 0$.

   (b) Use of theorem to write root in the form $a\sqrt{b}$, a rational and b a positive integer without square factors.

   Ex. $\sqrt{18}$, $\sqrt{98}$, $\sqrt{450}$, etc.

   (c) Use of commutative and associative properties of multiplication.

   Ex. $2\sqrt{57} \cdot 3\sqrt{38}$

   (d) Use of distributive property.

   Ex. $2\sqrt{3} + 5\sqrt{3}$; $5(2\sqrt{3} + \sqrt{5})$; $(\sqrt{5} + \sqrt{2})(\sqrt{5} - \sqrt{2})$

   (e) Radicals with variables.

   Ex. $\sqrt{3x^2}$. Since domain is the set of reals. $\sqrt{3}\ \sqrt{x^2} = \sqrt{3} \cdot |x|$. $\sqrt{x} \cdot \sqrt{x^3}$. Since domain is non-negative reals, $\sqrt{x^4} = x^2$, $x \geq 0$.

4. Square roots of rational numbers.

   (a) Theorem: $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$, $a \geq 0$, $b > 0$.

   (b) Use of theorem to simplify radicals (as defined above).

   Ex. $\sqrt{\frac{18}{25}}$, $\sqrt{\frac{7}{2}}$

   Also use variables: $\sqrt{\frac{25}{x}} = \frac{5\sqrt{x}}{x}$. Restrictions on x?

   $\sqrt{\frac{a}{b^2}} = \frac{1}{|b|}\sqrt{a}$. Restrictions on a? on b? on ab?

   $\sqrt{\frac{a^2}{b}} = (\frac{a}{|b|})\sqrt{b}$. Restrictions on a? on b?

6.5 Review of Properties of the Real Numbers and the Real Number Line.

1. Properties of the real number system

   Use exercise sets to recall field properties. (See Ex. 3, p. 279 of MJHS, Vol. 2)

   Include questions on subsets to emphasize properties not possessed by subsets. (See PFCA - H, p. 184).

   Emphasize order properties.

2. Real numbers and the number line.

   (a) Location of point for rational number $\frac{a}{b}$ by geometric construction.

   (b) Location of point for irrational $\sqrt{a}$ by geometric construction.

   (c) Location of point for infinite decimal by nested intervals. Use π as an example. (See MJHS, Vol. 2, pp. 256, 261-5)

   (d) One-to-one correspondence between real numbers and points on the line.

   (e) Exercises in ordering on the number line, numbers named by different kinds of numerals (fractions, decimals, radical absolute values).

GRADE 8 - CHAPTER 7

TRUTH SETS OF MATHEMATICAL SENTENCES

Background Assumptions:

Students have had some experience with the solution of some types of linear mathematical sentences with integral and rational coefficients. No formal methods of solution have been presented.

Students have had methods of solution of systems of mathematical sentences.

Rationale:

The development of a reasonably careful discussion of the solutions of mathematical sentences has been postponed until now because we wanted to discuss some operations which would not necessarily result in equivalent sentences, e.g., squaring both sides of a sentence, and multiplying both sides of an equation by an expression which is zero for some value or values of the variable.

Purposes:

1. To identify clearly the concept of equivalent mathematical sentences and to state precisely the "permissible" operations which will always lead to equivalent sentences.
2. To identify clearly the operations on mathematical sentences which may not lead to equivalent sentences.
3. To provide additional practice in problem analysis and problem solving techniques.

Procedure:

1. Review addition property of equality and its use in solving equations. Introduce the concept of equivalent sentences, p. 133-135. First Course in Algebra - Part I.

2. Review multiplication property of equality and its uses in solving equations. Use concept of equivalent equations. p. 167-170 - First Course in Algebra - Part I.

3. Apply these two properties to the solution of inequalities. Use concept of equivalent inequalities. p. 187-200 - First Course in Algebra - Part I.

4. Include appropriate verbal problems -- see above references to First Course in Algebra.

5. Consider the question of "permissible operations" for equivalent sentences, in general. p. 377-394 - First Course in Algebra - Part II.

6. Discuss the theorem - If $a = b$, then $a^2 = b^2$ and the fact that the converse is not true.

7. Consider equations of the following type:

   $$(x - 3)(x - 2)(x - 4) = 0.$$

   This may be written as the equivalent compound sentence --

   $$x - 3 = 0, \quad \text{or} \quad x - 2 = 0, \quad \text{or} \quad x - 4 = 0.$$

   This is another situation where equivalent sentences arise. p. 388-389 - First Course in Algebra - Part II.

8. Consider fractional equations and restrictions upon denominators containing variables. p. 391-394 - First Course in Algebra - Part II.

9. Consider the operation of squaring both sides of an equation and the fact that this operation does not always result in equivalent equations. A check is necessary to determine the solutions of the original equation. However, if boundary conditions are noted a logical check is not necessary, only a check for accuracy, e.g.,

   $$\sqrt{x} = 2 - x \quad \text{and} \quad 0 \leq x \leq 2.$$

   p. 394-398 - First Course in Algebra - Part II.

GRADE 8 - CHAPTER 8

QUADRATIC POLYNOMIALS AS FUNCTIONS

Background Assumptions:

1. Functions - Notation $f : x \rightarrow f(x)$; linear functions; graphs of linear functions.
2. The real number system, axioms and definitions, operations.
3. Solution of Mathematical Sentences
4. (Graphs of absolute value function)(See First Course, pp. 448-453)

Rationale:

The graph of $f : x \rightarrow x^2$ will be constructed. This will be compared with the graphs of the following:

$$x \rightarrow -x^2$$

$$x \rightarrow ax^2$$

$$x \rightarrow x^2 + k$$

$$x \rightarrow (x - h)^2$$

$$x \rightarrow a(x - h)^2 + k$$

In each case, the zeroes of the function will be discussed. This will lead to the discussion of how to find the zeroes algebraically. Since $ab = 0$ if $a = 0$ or $b = 0$, we would find it helpful to be able to factor quadratic polynomials.

Factoring, by use of the distributive property, of polynomials of the following types

$$ab + ac = a(b + c)$$

$$ax + ay + bx + by = a(x + y) + b(x + y) = (a + b)(x + y)$$

$$a^2 + 2ab + b^2 = a^2 + ab + ab + b^2 = (a + b)(a + b)$$

$$a^2 - b^2 = a^2 - ab + ab - b^2 = (a + b)(a - b)$$

will lead to the factoring of $x^2 + bx + c$, and of $ax^2 + bx + c$, by completing the square, and finally by inspection for those polynomials which can be factored over the integers.

Now we go back to use factoring in the finding of the zeroes of quadratic functions, and thus solving quadratic equations by factoring over the integers and by completing the square to factor over the reals. Finally we solve the general quadratic equation $ax^2 + bx + c = 0$ by completing the square, and thus develop the quadratic formula.

Factoring skills can also be put to work to rewrite $ax^2 + bx + c$ in the form $a(x - h)^2 + k$. This form is convenient for determining the minimum (or maximum) of the function, and thus for constructing its graph.

From a single graph of a function in form $ax^2 + bx + c$, the student can be shown how to find solution sets of many equations, by moving one (or both) of the axes.

Purpose:

1. To study in some depth the graph of the quadratic function.
2. To develop and practice the more common types of factoring of linear and quadratic polynomials.
3. To present methods of solving quadratic equations.

Procedure: Ref: First Course, pp. 533-536

Section 1. Graph of the quadratic function.

1.1 Graph of $f : x \rightarrow x^2$, and of $f : x \rightarrow -x^2$.

Do graph of $x \rightarrow x^2$ carefully for $-4 \leq x \leq 4$ - include $(\frac{1}{2}, \frac{1}{4})$, $(\frac{1}{4}, \frac{1}{16})$ to show the shape of the parabola.

Point out: symmetry with respect to y-axis; range is non-negative reals; domain is set of reals.

Show graph of $x \rightarrow -x^2$ as a reflection about the x-axis; symmetry is preserved.

1.2 Graph of $f : x \rightarrow ax^2$.

Consider graphs of $x \rightarrow ax^2$ for $0 < a < 1$ and for $a > 1$, and compare with graph of $x \rightarrow x^2$.

Show graphs of $x \rightarrow ax^2$ for $-1 < a < 0$ and for $a < -1$ as reflections of the corresponding cases above.

Perhaps generalize on $x \rightarrow ax^2$ for $a \neq 0$, in terms of $|a|$.

1.3 Graph of $f : x \rightarrow ax^2 + k$.

Discuss effect of k for $k > 0$ and $k < 0$.

For $k < 0$, consider the zeros of the function (new term--explain it!)

Ask what about zeros for $k = 0$, and $k > 0$. (This hints at complex numbers)

1.4 Graph of $f : x \rightarrow a(x - h)^2$.

Discuss effect of h for $h > 0$ and $h < 0$.

Consider the zeros in both cases.

1.5 Graph of $f : x \rightarrow a(x - h)^2 + k$.

Summarize how the graph of $x \rightarrow x^2$ is affected by values of a, h, and k.

Talk about zeros of the function, also maximum or minimum, symmetry, turning point.

1.6 Two algebraic questions:

(1) How can a quadratic function in form $ax^2 + bx + c$ be rewritten in form $a(x - h)^2 + k$, to make graphing simpler, as well as to aid in locating line of symmetry, turning point, and maximum or minimum value. (Merely develop feeling of a need for factoring skills of some sort.)

(2) How can the zeros of a function in form $ax^2 + bx + c$ be determined algebraically? Since $ab = 0$ if and only if $a = 0$ or $b = 0$, this calls for being able to factor the polynomial over some set of numbers.

Section 2. Factoring Polynomials. (9H, 556-572, 577-588, 591-621)

2.1 Meaning of factoring over the integers, over the rationals, over the reals.

2.2 Type $ab + ac = a(b + c)$.

Simple use of distributive property.

2.3 Type $ax + ay + bx + by = (a + b)(x + y)$.

Multiple use of distributive property.

2.4 Perfect square, $a^2 + 2ab + b^2$.

Show use of distributive property, as:

$$a^2 + 2ab + b^2 = a^2 + ab + ab + b^2 = a(a + b) + b(a + b) = (a + b)^2$$

Discuss characteristics of a perfect square trinomial.

2.5 Difference of squares, $a^2 - b^2$.

Show use of distributive property, as:

$$a^2 - b^2 = a^2 - ab + ab - b^2 = a(a - b) + b(a - b) = (a + b)(a - b)$$

Then observe short cut, from reversing the product:

$$(a + b)(a - b) = a^2 - b^2.$$

Move on to types:

$$(a + b)^2 - c^2 = (a + b + c)(a + b - c)$$

$$a^2 + 2ab + b^2 - c^2 = (a + b)^2 - c^2 = (a + b + c)(a + b - c)$$

2.6 Factoring a polynomial of form $x^2 + bx + c$ by completing the square.

Also show by inspection for those factorable over the integers.

2.7 Factoring a polynomial of form $ax^2 + bx + c$ by completing the square, and by inspection for those factorable over the integers.

Section 3. Solving quadratic equations.

3.1 Discuss "finding zero of function $ax^2 + bx + c$" as "solving equation $ax^2 + bx + c = 0$".

3.2 Solve by factoring; i.e., $ab = 0$ if $a = 0$ or $b = 0$.

Emphasize factoring by completing the square as the general method, giving roots both over the rationals and over the reals.

Hint at complex roots.

3.3 Development of formula:

Generalization of completing the square.

Emphasize its significance in that it relates the roots to the coefficients.

Section 4. Going from $ax^2 + bx + c$ to $a(x - h)^2 + k$.

4.1 Use of completing the square to write $ax^2 + bx + c$ in form $a(x - h)^2 + k$ and thus draw the graph quickly and accurately.

4.2 From the graph of a single function of form $ax^2 + bx + c$, the solutions of many quadratic equations can be found by giving the function a succession of values.

GRADE 8 - CHAPTER 10

PARALLELS AND PERPENDICULARS

Background:

Separate chapters have been introduced on parallelism (in Grade 7) and perpendicularity (in Grade 8). The present chapter summarizes, reviews, synthesizes and extends these chapters.

From work with measure we need the concept that the length of a line segment and the distance between two points are the same number arrived at in different ways. The first is a function from the set of segments in space to non-negative real numbers; the second is a function from pairs of points to non-negative real numbers. We are now ready to extend the concept of "distance between" to a function whose domain is any pair of geometric figures. (There is still a problem to straighten out here! Will we accept a line segment of length 0? How about "two" points which have a distance of 0 between them?)

Many properties of specialized quadrilaterals are assumed to be already known and are summarized here. Some simple reflections in points, lines, and planes as transformations are used as background for discussing symmetries of 2 and 3 dimensional figures. The concept of symmetry of a figure is then made a little more general.

Although the parallel "property" has been introduced and intuitive facts associated with it, we incorporate it here into a deductive sequence.

Purpose:

The purposes of this chapter are as follows:

1. To consider sets of parallel and perpendicular lines and planes and the number of regions they determine in 2 and 3 space.

2. To develop deeper intuition in 3-space for the relations of parallel and perpendicular among lines and planes.

3. To define distance between parallel lines and planes, but within the broader context of distance between two geometric objects.

4. To survey and extend concepts related to quadrilaterals.

5. To lay a simple and intuitive foundation for the concept of necessary and sufficient conditions.

6. To extend the concept of reflections in point, line, and plane to symmetries of polygons.

7. To provide one more short deductive sequence, this time applied to parallel lines.

Rationale:

Counting regions is a way of associating numbers with geometric figures. In a sense it is a function from a certain set of geometric figures to the natural numbers. This use of number is not a metric use, but a combinatorial use of numbers for counting.

Parallel and perpendicular lines and planes taught separately are rich subjects, but in this chapter we consider the richer interconnections between these two relations.

Distance up to now is a function on a pair of points, now it is extended and generalized, still including former ideas as special cases.

Symmetries are very definitely considered here as one more step along the road which stretches from reflections (in Grade 7) to transformations (in Grade 9). There are symmetries that are not reflections; later there will be transformations that are not symmetries.

Many times we have used the fact that the degree measure of the angles in a triangles add to 180, but here, for the first time, we prove this fact in a miniature, deductive system.

Section 1. Regions:

1.1 Study number of regions (not counting the lines) into which a plane is separated by two parallel lines; by n parallel lines.

1.2 Study number of regions (not counting the lines) into which a plane is separated by two perpendicular lines; by two parallel lines and transversal perpendicular to them; by net of n lines and m others perpendicular to them.

1.3 Extend ideas of 1.1 and 1.2 carefully to some problems in 3-space with parallel and perpendicular planes.

Exercises:

1. Use sequence of simpler problems to reach such problems as these:
   (a) How many regions of a plane are formed by 4 parallel lines and 5 lines perpendicular to them? (Ans: 30)

   (b) How many regions of a plane are formed by n parallel lines and by m lines perpendicular to them? Ans: $(n + 1)(m + 1)$

2. Just a few suggestions for problems in 3-space:

   (a) α, β, and γ are planes so that $\alpha \| \beta$ and $\alpha \perp \gamma$. Into how many regions do these separate space? (Ans: 6)

   (b) α, β, and γ are 3 mutually perpendicular planes. Into how many regions do they separate space? (Ans: 8)

   (c) Into how many regions is space separated by n parallel planes? (Ans: $n + 1$)

3. Perhaps some challenge problems of this level, but not any harder:

   (a) What is the maximum number of regions you can create in a plane with 3 lines? with 4 lines? with 5 lines?

   (b) What is the maximum number of regions you can create in space with 2 planes? with 3 planes? with 4 planes?

Section 2. Combining Parallel and Perpendicular Relations:

2.1 Line perpendicular to one of two parallel lines; line perpendicular to one of two parallel planes.

2.2 Two lines perpendicular to same line; two lines perpendicular to plane.

2.3 Plane perpendicular to one of two parallel lines; plane perpendicular to one of two parallel planes.

2.4 Two planes perpendicular to same line; two planes perpendicular to same plane.

2.5 Consider relations of parallel and perpendicular with respect to reflexive relations, symmetric relation, and transitive relation. (We assume that these terms have been introduced in connections with numbers, so we are making use of an old idea here. They are intended to reinforce and review such an idea, not to introduce it.)

Exercises:

1. More ASN exercises in 3-space: (The problem of whether a line should be considered parallel to itself; a line in a plane parallel to the plane; and a plane parallel to itself might well be considered by some other group of people, and by the writers. Here the viewpoint is that two lines must be distinct in order to be parallel, and so on.)

   A S N (1) Hypothesis: Two planes are parallel.
   Conclusion: A line perpendicular to one of these planes is perpendicular to the other.

   A S N (2) Hypothesis: Two lines are parallel.
   Conclusion: A plane perpendicular to one of these lines is perpendicular to the other.

A S N (3) Hypothesis: Two planes are parallel.
Conclusion: A plane perpendicular to one of these planes is perpendicular to the other.

A S N (4) Hypothesis: Two lines are parallel.
Conclusion: A line perpendicular to one of these lines is perpendicular to the other.

A S N (5) Hypothesis: Two planes are perpendicular.
Conclusion: A line perpendicular to one of these planes is perpendicular to the other.

A S N (6) Hypothesis: Two planes are perpendicular.
Conclusion: A line perpendicular to one of these planes is parallel to the other.

A S N (7) Hypothesis: Two planes are perpendicular.
Conclusion: A line parallel to one of these planes is perpendicular to the other.

A S N (8) Hypothesis: Two lines are perpendicular.
Conclusion: A plane parallel to one of these lines is perpendicular to the other.

A S N (9) Hypothesis: Two lines are perpendicular.
Conclusion: A line perpendicular to one of these lines is parallel to the other.

A S N (10) Hypothesis: A plane is perpendicular to a line.
Conclusion: Another plane perpendicular to the line is parallel to the first plane.

A S N (11) Hypothesis: Two planes are perpendicular.
Conclusion: A plane parallel to one of these planes is parallel to the other plane also.

2. Fill in the following table with the letter F or T with the following meanings: (Assume we are in 3-space.)

T The relation has the property.

F The relation does not have the property.

	Reflexive	Symmetric	Transitive
$a \| b$			
$a \perp b$			
$\alpha \| \beta$			
$\alpha \perp \beta$			

Note: a and b are lines; α and β are planes in this standard notation; all lines or planes considered in the transition relation are distinct.

Section 3. Direction of a Plane.

3.1 What do a family of parallel lines in 2-space have in common: slope.

3.2 What do a family of planes in 3-space have in common: the slope of a plane is difficult to define; go back to the plane for inspiration.

3.3 Another characteristic of a set of parallel lines in 2-space: there is a line perpendicular to all the parallel lines; go back to 3-space and see if this helps.

3.4 Hurrah: For each set of parallel planes there is a line perpendicular to all planes (this does not imply only one line).

3.5 Discuss "characterization" of a family of parallel lines by the slope of the line found perpendicular to them ; if we could get to the idea of slope of line in 3-space we could do the same -- perhaps in the future.

Exercises:

Have students draw diagrams to illustrate the ideas above.

Section 4. Distance between Parallel Lines and Parallel Planes.

4.1 Review: Distance from point to point is length, or is measure of segment.

4.2 Def: Distance from point to set of points as minimum of distances to points in the set; apply to line, segment, circle, and plane. (The problem of unusual sets where the idea of a greatest lower bound is needed should be avoided. Just simple cases here.)

4.3 Def: Distance from set to set as minimum of distances from point in one set to point in other set; apply to line and circle, two circles (completely outside or one contained in other), two parallel lines, two parallel planes, two intersecting lines.

4.4 Def: "the" altitude of a parallelogram or of a trapezoid; altitude as segment and as number.

4.5 Equations of planes parallel to coordinate planes; inequalities for "strips" and 3-space intervals. (See previous work in Section 2.6 of Grade 7, Chapter 11, Parallelism)

Typical Exercises:

(In the following discussion note the difficulty in keeping pure meanings for the phrases "distance from A to B" and "distance between A and B". More thinking must be done to say when we want to talk exactly, and when we may use colloquial expressions in these situations.)

1. A point is 6 inches from the center of a 3-inch circle. How far is it from the circle.
2. A point is 2 inches from a circle which has a radius of 3 inches. How far is the point from the center of the circle.

3. A point lies on the perpendicular bisector of a 4-inch segment and is 3 inches from the segment. It moves parallel to the segment through 6 inches. How far is the point now from the segment?

4. A point is at the center of a 5-inch circle. This point moves 12 inches in a direction perpendicular to the plane of the circle. How far is the point now from the circle? Now the point moves 5 inches parallel to the plane of the circle; how far is it now from the circle? If the point had moved 21 inches instead of 5, how far would it have been from the circle?

   (More thinking needs to be done about the use of such expressions as "The point moves parallel to a line", or "The point moves parallel to a plane". The meaning is clear, but how should we say it in mathematics? Can a point move or is it not fixed in space? Should we not say "moves on a line which is parallel to the line."? These indicate some of the difficulties.)

5. A 6-foot flagpole is attached to the vertical wall of a building at a point 20 feet above the horizontal street. The pole makes an angle of 45° with the horizontal. How far is each end of the pole from the street?

6. A line is 6 inches from the center of a 2-inch circle. How far is the line from the circle?

7. Two circles have radii of 5 and 10 inches. What is the distance between the circles if:

   (a) Their centers are 20 inches apart?
   (b) Their centers are 2 inches apart?
   (c) Their centers are 6 inches apart?

8. Two sides of a parallelogram are 6 inches and 12 inches and the angle included between them is 30°. What is the distance between the pairs of parallel lines containing opposite sides? Is this the same as the distances between opposite sides (as segments)?

9. Compute the area of the parallelogram in Exercise 8 in two ways and compare your answers.

10. In 3-space what is the graph of the following sentences:

    (a) $4 \leq x \leq 7$

    (b) $4 \leq x \leq 7$ and $-3 \leq y \leq 2$

    (c) $4 \leq x \leq 7$ and $-3 \leq y \leq 2$ and $6 \leq z \leq 8$

11. What is the distance between two intersecting lines?

Section 5. The Quadrilateral Properties:

5.1 Review, summarize and interrelate properties of parallelogram, trapezoid, rhombus, rectangle and square; properties of sides, diagonals and angles are intended.

5.2 Study the sufficiency and begin the idea of necessity of conditions on a very intuitive level; necessary and sufficient conditions as such to be studied later.

Typical Exercises:

1. The following problem is a large one and may not be wise to include as a whole, but it is extremely valuable. It would be excellent summary material and might well be worth 2 or 3 days of class time.

   Directions: In the following table, consider the five types of geometric figures listed across the top with respect to the sixteen statements at the left (to save space the statements are referred to by number and listed below). If the geometric figure at the top always has the property at the left, fill in the table with an A; if the figure sometimes has the quality, use an S; and if it never has that property, use an N.

	Rectangle	Square	Rhombus	Parallelogram	Trapezoid
1.					
2.					
3.					
etc.					

Statements:

1. Both pairs of opposite angles are congruent.
2. Both pairs of opposite sides are congruent.
3. Each diagonal bisects two angles.
4. The diagonals bisect each other.
5. The diagonals are perpendicular.
6. Each pair of consecutive angles is supplementary.
7. Each pair of consecutive sides is congruent.
8. Each pair of consecutive angles is congruent.
9. The diagonals are congruent.
10. Both pairs of opposite sides are parallel.
11. Three of its angles are right angles.
12. Its diagonals are perpendicular and congruent.
13. Its diagonals are perpendicular bisectors of each other.
14. It is equilateral.
15. It is equiangular.
16. It is both equilateral and equiangular.

2. Another, valuable use of this table is to turn the problem around and thus ask the following question: Are the data at the left of a line sufficient to assume the figure is the type named at the top? Use the letters A, S, N to mean always, sometimes and never as follows:

A The data being true, the figure must be that given.

S The data being true, the figure may or may not be that given.

N The data being given, the figure could never occur.

Section 6. Symmetries:

6.1 Def: Symmetry in a line, in 2-space, in 3-space.

6.2 Def: Symmetry in a point, in 2-space, in 3-space.

6.3 Def: Symmetry in a plane, in 3-space.

6.4 Symmetries of triangles: isosceles, equilateral.

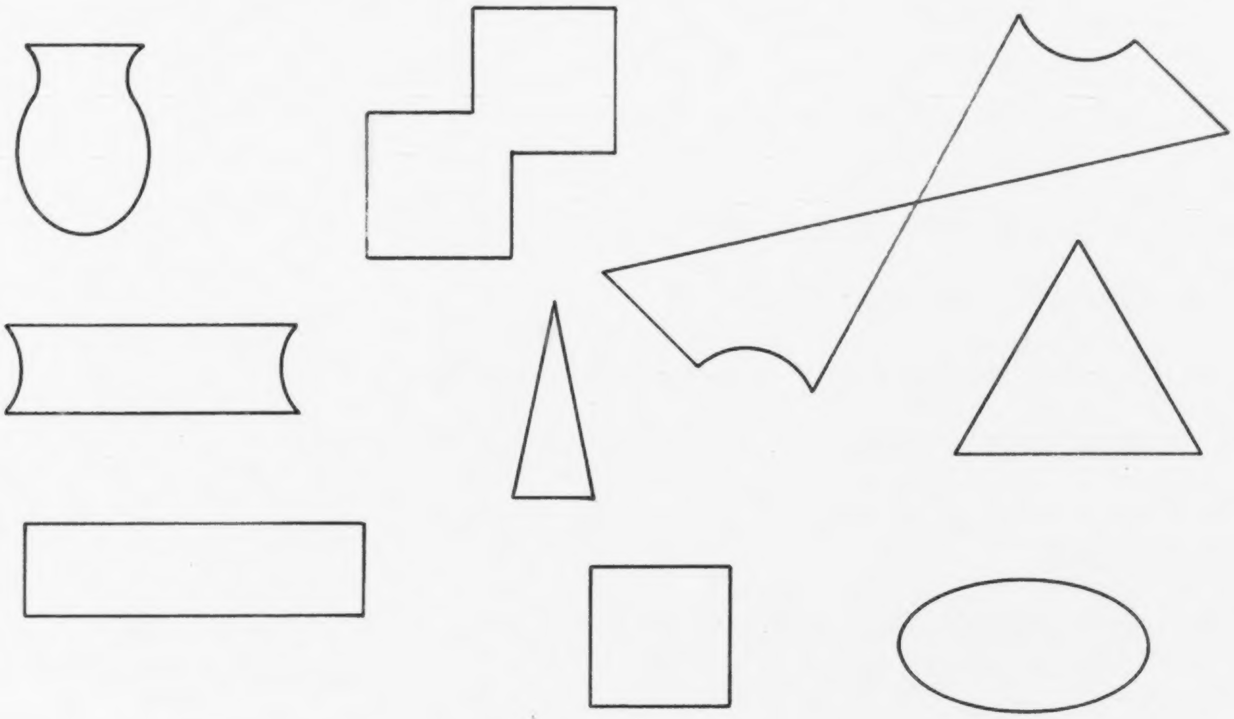
6.5 Symmetries of rectangles: non-square, square.

6.6 Symmetries of a circle.

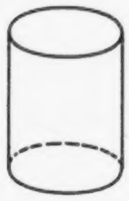
6.7 Symmetries of three dimensional figures.

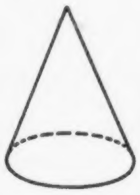
Exercises:

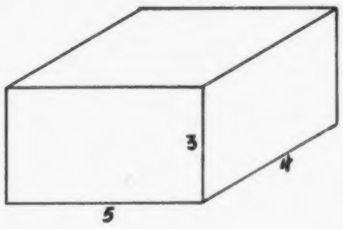
1. For each of the following figures use a ruler and draw all lines of symmetry. Then write down the number of lines you have found. Then mark each point of symmetry you can find.

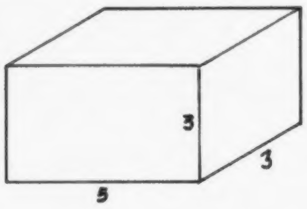


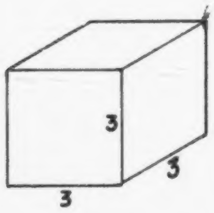
2. Tell for each figure how many planes of symmetry there are; then how many lines of symmetry; then how many points of symmetry.

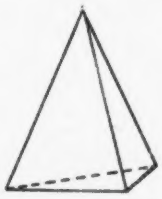




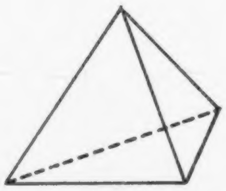




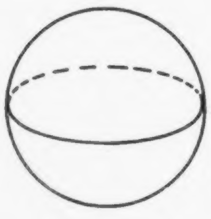




Equilateral Base

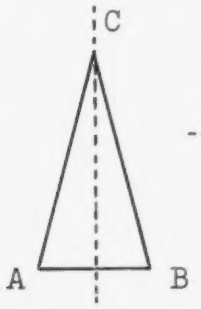


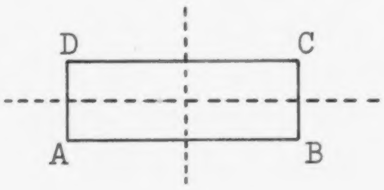
Regular Tetrahedron

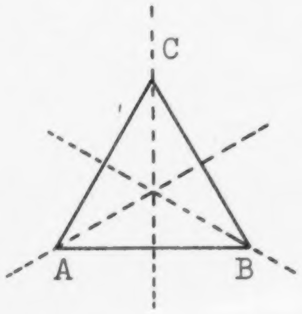


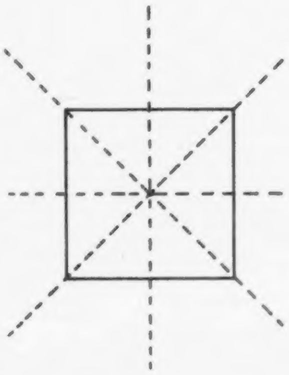
Sphere

3. For each of these figures find reflections in all lines of symmetry; then rotations about center of 180°, 120°, 90°, or whatever you need to make the figure coincide with itself.









4. Symmetries of a circle, rotations about the center only.

Section 7. Angle - Sum Proofs.

7.1 The parallel property.

7.2 Proof: Angle measure sum for triangles.

7.3 Semi-proof: Angle sum measure of convex polygons.

Exercises:

The usual kinds. See SMSG and other books.

GRADE 8 - CHAPTER 11

PROPERTIES AND MENSURATION OF GEOMETRIC FIGURES

(Review and Summary)

Background:

Metric System, Grade 4, Part II, Chapter 9, pp. 476ff.

Unit Segments, Unit Angles, What is Area?, Grade 5,
Part II, Chapter 7, pp. 407ff.
Chapter 8, pp. 455ff.

Congruence, Grade 7, Chapter IV

Measure, Grade 7, Chapter V.

Purpose:

1. Review and extend notion of measure; develop some degree of comfort with the metric system.
2. (a) Based on the summary in Chapter 10, Section 5 review and extend formulas for perimeters and areas and apply to problems involving real numbers. Derive the formula for the area of a trapezoid.

   (b) Deepen the understanding of the relation between congruence and measure: (congruence $\Rightarrow$ equal measure, converse is not true).
3. Review properties of regular polygon. Compute perimeter, radius, apothem, area. Develop (WST) formula for area of circle.
4. Develop properties of solids:
   (a) with parallel bases: boxes, cylinder.
   (b) pyramid, cone

(c) develop formulas for surfaces and volumes of solids.

(d) sphere (WST).

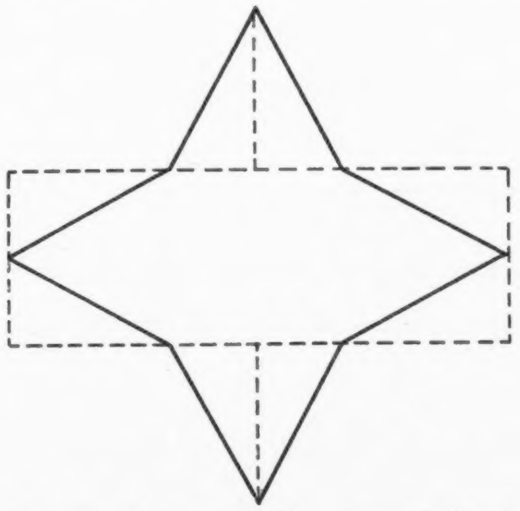
OUTLINE

(1) This is the fourth exposure to measure and it is important to continue this development, building on previous learning experiences.

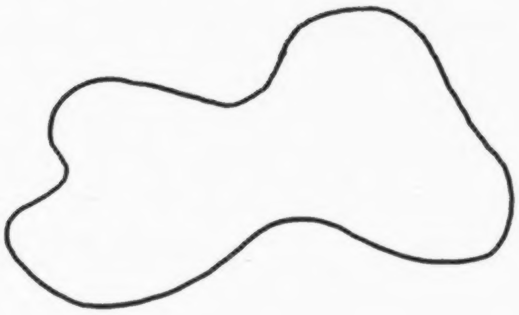
First, measure was based on congruence: Two segments are congruent if one is an exact copy of the other; the same holds for regions.

Next: linear measure was given a numerical value.

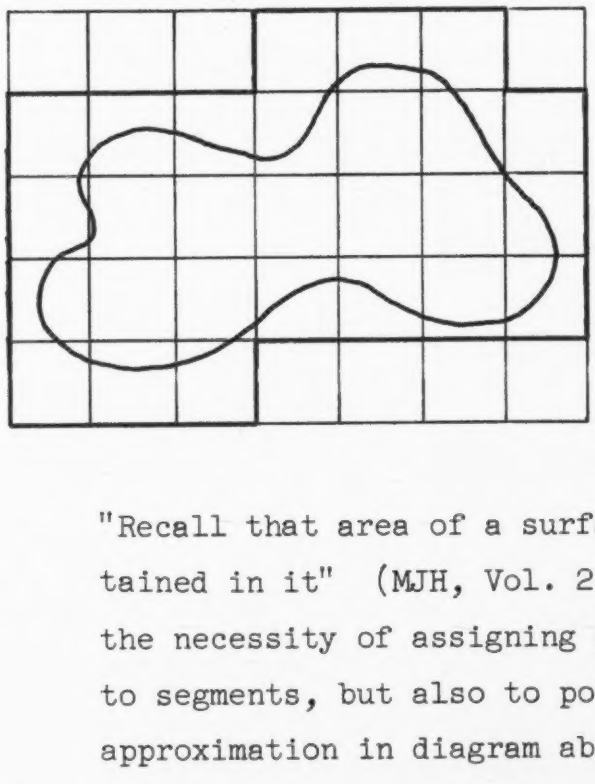
The third time, in Chapter V, Grade 7, area is developed in terms of equivalent regions. This section contains excellent examples; review some, bring in new ones. Although we are generally concerned with convex polygonal regions, an example could be introduced to show that not all regions are convex, yet can be decomposed as indicated.

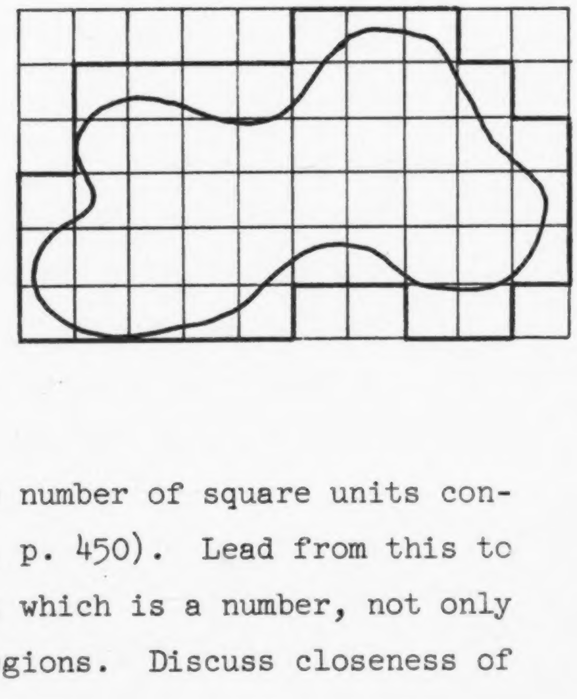


But now we want more; we want a numerical measure for areas. Show that not all regions can be decomposed easily into equivalent regions. For example:



It would not be possible to decompose this region equivalent to another region. But we can try the following in order to estimate the area:



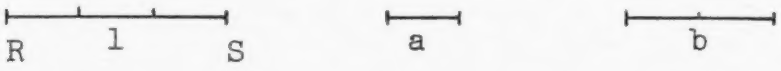


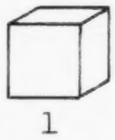
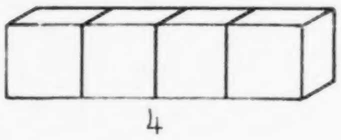
"Recall that area of a surface is the number of square units contained in it" (MJH, Vol. 2, Part II, p. 450). Lead from this to the necessity of assigning a measure, which is a number, not only to segments, but also to polygonal regions. Discuss closeness of approximation in diagram above.

Develop next the idea of arbitrary unit versus standard unit by examples:

1. (a) Draw $\triangle ABC \sim \triangle DEF$ with ratio of similitude $\frac{1}{2}$. $\overline{AB}$, $\overline{BC}$, $\overline{CD}$ are arbitrary and represent units on which $\overline{DE}$, $\overline{EF}$, and $\overline{FD}$ are based.

(b) If segment RS has measure 1 then $a = \frac{1}{3}$, and $b = \frac{2}{3}$, since a is contained three times in the given segment and twice in b.



Similarly, if  then  ,

(c) Recipe for cooking: The weight of an egg could be the unit.

(d) Pacing off distances: Length of step is the unit.

(e) Game: Go forward until I count to five; unit of time; interval from 1 to 2, etc.

Next, ask questions of this type: Can you choose the units when paying for your lunch? weighing yourself? reading a map? buying gasoline?

2. Include questions of the type: What would be a reasonable unit of measure to express:
   (a) The distance from your home to school.
   (b) The height of a telegraph pole.
   (c) The length of a desk.
   (d) The depth of a bookshelf.
   (e) The length of the hands of a wrist watch.
   (f) The length (?) of periods in a school day.
   (g) The weight of a beam for a bridge.
   (h) The weight of flour in a cake.
   (i) The amount of milk in a cake.

Depending upon Chapter 5, Grade 7, and the degree of detail given to metric measure, review and complete metric measures for distances, areas, volumes and weights. Include here or at the end of the text a table with metric measures.

3. Raise question. Can you give the corresponding metric measures in 2. above?

Point out simplicity of U.S. monetary system compared to British system. In the same way, compare linear measures mm, cm, dm, m, etc., with inch, foot, yard, etc.

4. Include questions of the type:
   (a) Which is longer, an 8 inch or 12 cm pipe?
   (b) If the speed limit on a highway is 60 mph., does a driver break the law if he travels at 90 km per hour?
   (c) Would you expect to pay more for 3 quarts of milk or for 3 liters of milk?

It may be preferable to use measure for areas, volumes, weights, etc., as (2) and (3) in this outline are first presented. But since the students are probably familiar with these ideas a review here and application later in the chapter ought to work out successfully.

2(a) Review the formulas for perimeter and area of parallelogram and triangle and develop the formula for the area of the trapezoid. (See MJH, Vol. 2, Part II, Chapter 11, pp. 451-455)

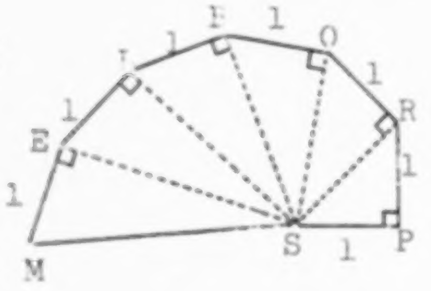
Next refer to the statement in Chapter 5, Grade 7, page 2: ". every line segment has a measure. Tread (but lightly) on the idea that this measure often is a number familiar to them, but that the measures of many line segments are numbers which they have not studied yet. Until they meet these numbers, they can only give such measures by approximation." (NOTE: there are approximations of different natures involved.)

Now, we are ready to assign the exact measure to segments, the interior of plane figures, surfaces of solids, etc. The students have studied real numbers. There are times when rational approximations are important. Refer back to Chapter 5 and see how good the approximations were at that time, when only rational numbers were known to the students.

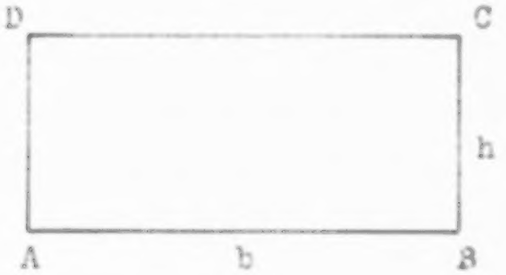
Use examples as on page 453 (reference given above), finding the length of diagonals and perimeters. Other examples:

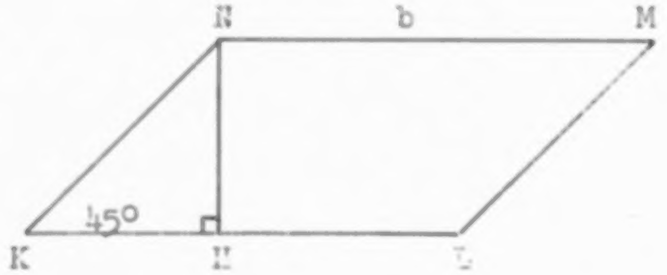
1. (a) Find the perimeter of the polygon PROBLEMS.

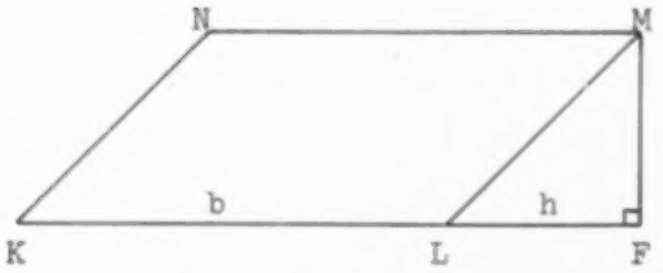
   (b) Can you draw a polygonal region like "PROBLEMS", whose perimeter is 12 units?

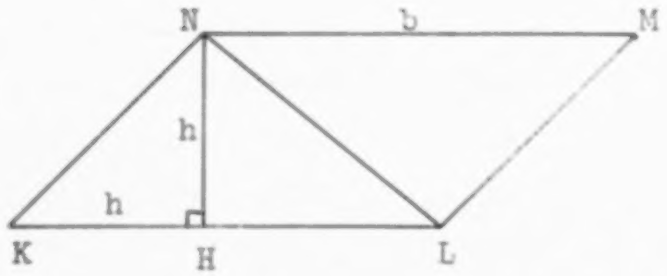


2. A rectangle and a parallelogram have equal base and height. One angle of the parallelogram has a measure of 45°. How do the diagonals of the two figures compare? their perimeters? their areas? (Emphasize the role of a good diagram). First assign numerical values to h and b, then generalize.



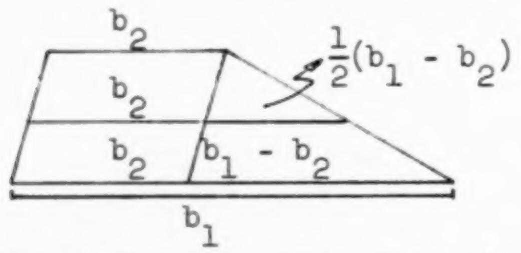






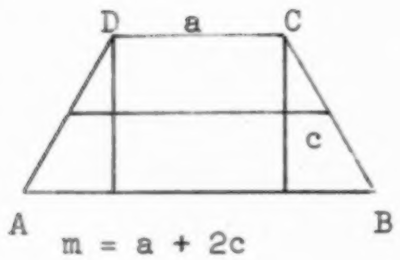
Repeat with $m\angle K = 60^\circ$; $m\angle K = 30^\circ$.

3. Compute the length of the median of a trapezoid: assign definite values to b_1, b_2, then generalize.

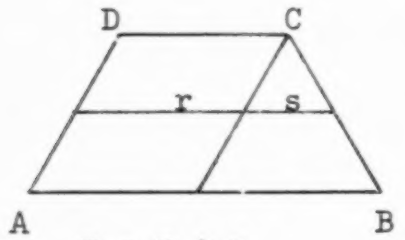


Use similar triangles, develop formula $\frac{1}{2}(b_1 + b_2)$. Now, derive area in different way.

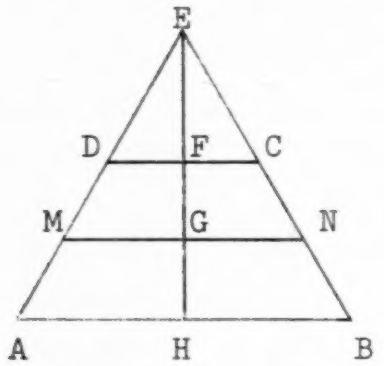
Repeat for isosceles triangle and use different ways suggested by the diagrams.



$m = a + 2c$



$m = r + s$

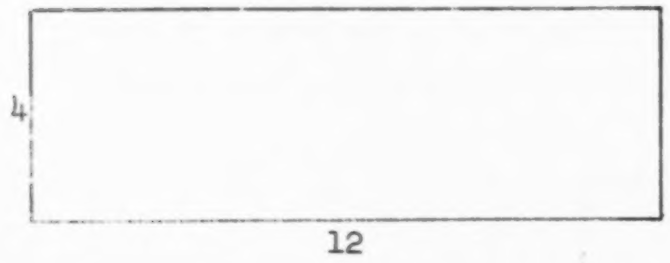


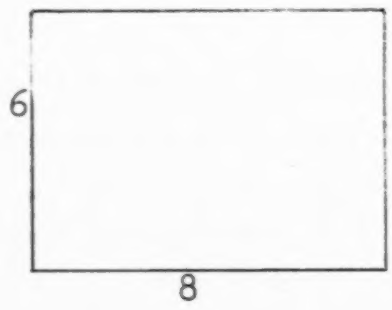
$EH = 2FH$ Use ratios to find MN.

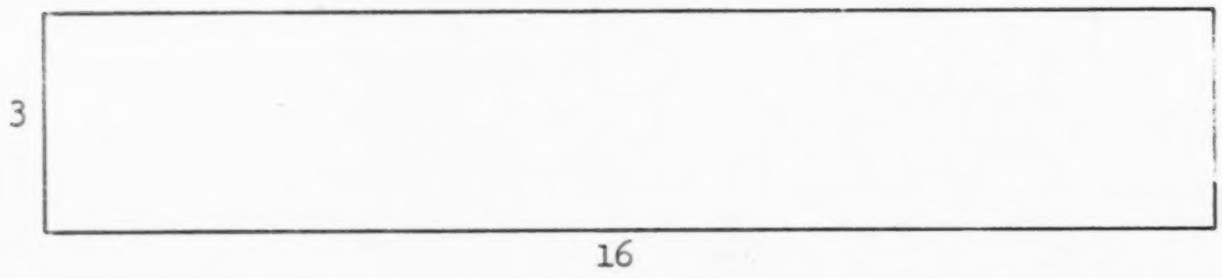
2(b) Measure is based on congruence. Use examples that extend the idea from segments and plane figures to 3 dimensions: Boxes, can goods: identical containers, equal contents, equal weights, equal prices (?).

Example 1:

In contrast: the area of a rectangle is 48 square inches. Are the following rectangles congruent?



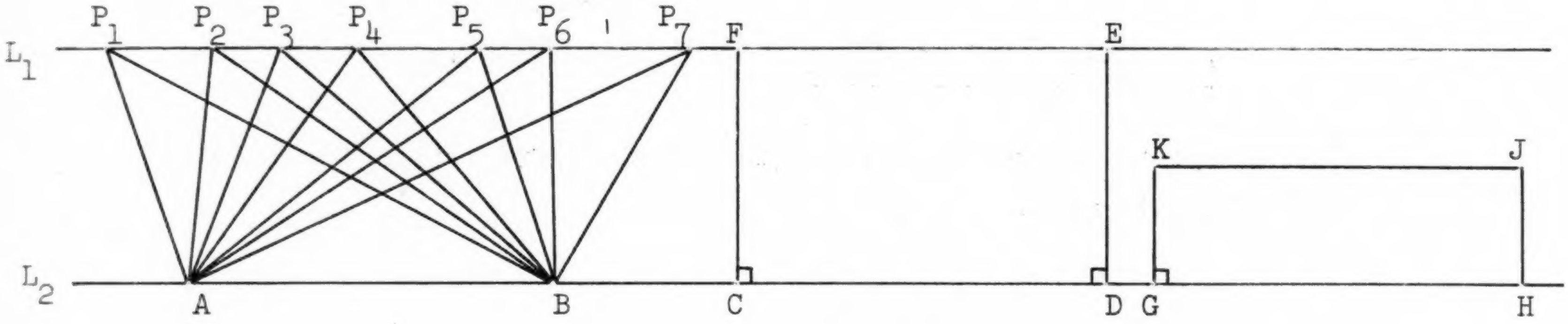




Example 2:

Given $L_1 || L_2$, $\overline{AB} \cong \overline{CD} \cong \overline{GH}$ and $ED = 2JH$

[Authors: do not use subscripts]



(a) The length of which segments is the distance between L_1 and L_2? Call this distance h.

(b) What have the triangles in common? (You will get different answers.) If $AB = b$, what is the area of $\triangle ABP_1$? What is the area of triangle ABP_2, ABP_3, ABP_4, ABP_5, ABP_6, ABP_7?

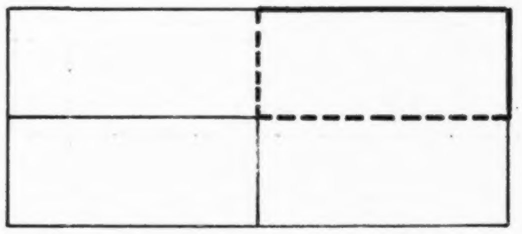
(c) Are any of these triangles congruent?

(d) Could you find a point on L_1, say $Q \neq P_1$, such that $\triangle AQB \cong \triangle AP_1B$?

(e) How do the areas of the triangles compare to the area of rectangle CDEF?, rectangle GHJK?
Formulate conclusions from the above.

Develop the understanding of the formula by using both geometric and algebraic approach:

Example: What is the effect upon the perimeter of a rectangle if the length and the width are doubled? What is the effect upon the area?



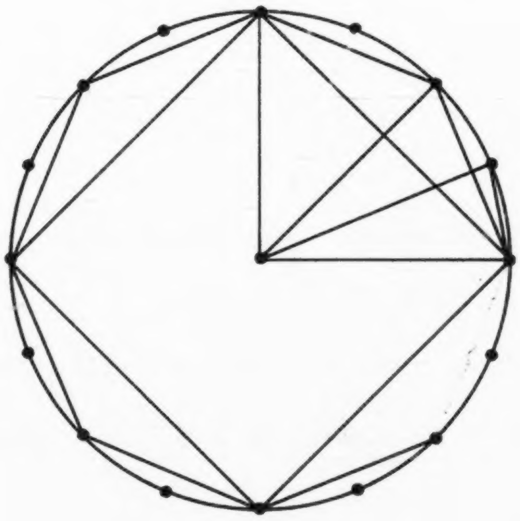
$$\begin{aligned} p_1 &= 2a + 2b = 2(a + b) & A_1 &= ab \\ p_2 &= 2(2a) + 2(2b) & A_2 &= (2a)(2b) \\ &= 2(2(a + b)) & &= 4(ab) \\ &= 2p_1 & &= 4A_1 \end{aligned}$$

The student should recognize when there are linear factors, quadratic factors; algebra and geometry should support one another.

(3) Review properties of regular polygons. Introduce or review the vocabulary: center, radius, chord, apothem, circumscribed circle, inscribed polygon.

Compute p, given r and a
r, given a and s
a, given r and s

Let the student visualize by making their own diagrams, what happens to the perimeter of an inscribed regular polygon, as the number of sides goes from 4 to 8 to 16, etc. Then, show with geoboard: nails, equidistant on the circle, rubber bands forming regular n-gons.



Compare p to c and area of interior of Δ to sector as n increases.

Let them measure the perimeters and record results. Have each student cut out a disk of different size, from stiff paper, or use any circular object available. Have them measure the diameter and the circumference by wrapping a string around; have them compare the ratio $\frac{c}{d}$. Exchange the circular objects and try it again. What is the value of this ratio? How do these values compare? Since it ought to be close to 3 but (hopefully) is not exactly 3, it is some number, let's call it π.

So, if $\frac{c}{d} = \pi$, $c = \pi d$ or $c = 2\pi r$, $\pi \approx 3.14$. How do the perimeters obtained before, compare? Is $\frac{p}{2r} \approx 3$?

[See MJH, Vol. 1, Part II, pp. 490-500.]

See whether the relation, informally and experimentally only, of course, can be "discovered": as n becomes very large, $p \rightarrow c$.

Observe at the same time that the interior of the polygon comes closer and closer to the interior of the circle.

The transition from $A = \frac{1}{2}ap$ to $A_O = \pi r^2$ is then a simple step. Again: as $n \rightarrow \infty$, A polygon $\rightarrow A_O$. [See MJH, Vol. 1, Part II, 11-7, p. 500.]

(4) Assemble as many models of solids as possible. Where do we see, indoors and outdoors, in a store, factory, etc., objects like some of these?

Analyze the properties and let students "discover" some way of classification.

Introduce the terms: face, edge, vertex, surface, diagonal, volume.

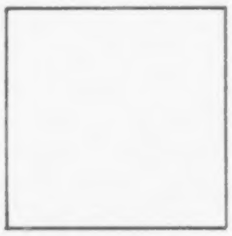
Have the student think about a wire model, a cardboard model before "dissecting" the solids.

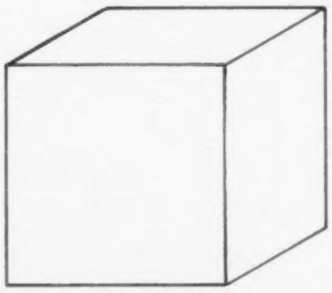
Let them count the number of vertices, faces and edges and record in a table. Can they discover Euler's formula, $V + F - E = 2$?

Let the student see what plane contains a diagonal in a cube or prism. What dimensions must be known to determine others.

Just as the length of a segment is the number of units contained in it and the area of a surface is the number of square units contained in it, introduce the volume of a solid as the number of cubic units contained in it.

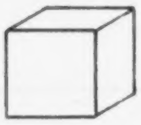
1 inch


1 sq. inch


1 cubic inch

1 cm

1 sq. cm


1 cubic cm

Now group solids according to:

(a) Parallel bases: Prisms and cylinder; boxes with squares, rectangles, triangles as bases; cylinder with circles as bases (of course, the interior of these polygons is meant); $V = Bh$.

(b) Pyramids and cones. $V = \frac{1}{3} Bh$. [See MJH Vol. 2, Part II, pp. 465-489].

(c) Compile a list of formulas as a result of the above.

Students should make models of as many solids as possible. Have different students use different dimensions, but prisms and pyramids with congruent base and height should be made, similarly for cylinder and cone, [see MJH, Vol. II, Part II, pp. 491-510] so that volumes can be compared and experimented with.

There is an excellent opportunity to apply real numbers, similarity and design problems from "real life".

Example: A container must have a certain capacity. Should it be made of tin or with tin tops and cardboard sides? Which is more economical? Which shape, cubic, cylindrical or rectangular? Etc.

The Sphere (WST)

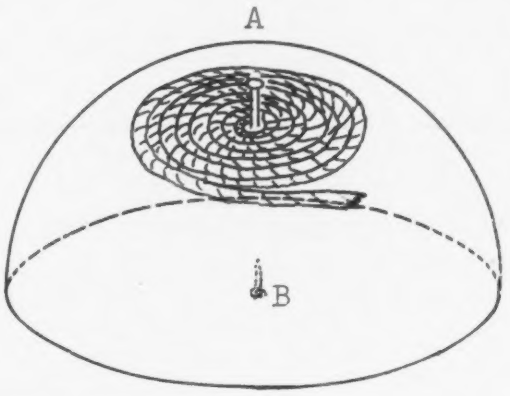
(4d) (1) Start with The Earth as representation of the sphere.
Use parts of MJH, Vol. 2, Part II, pp. 511-529.

(2) Surface of sphere.

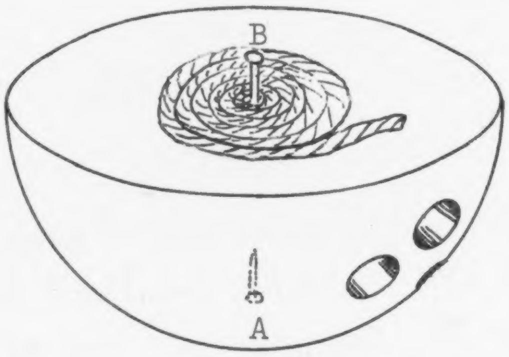
Experiment: Take a bowling ball -- hopefully, an eighth grader can get one -- and saw it in half. Put two nails in, as shown in diagram. Get a ball of heavy string.

1. First: Tie the string at A and wind around without slipping, until the total hemisphere is covered. Mark the end of the string by a knot.

   Next, take a guess: how much of the string used above will be needed to cover the interior of circle B? Write your guess on a piece of paper.



2. Now tie, as before at A, the string at B and carefully cover the circular region. How much of the string used in 1 did you use this time? How does your guess compare with the experiment?



Students may be able and willing to try the experiment with some other spherical object. There is hope that the result will be close to: the surface of the hemisphere is twice that of the circular region or

$$S_H \text{ sphere} = 2\pi r^2$$

or

$$S \text{ sphere} = 4\pi r^2$$

(3) Volume of sphere.

Use 12-5, MJH, Vol. 2, Part II, pp. 533ff.

OR:

The volume of the sphere can be though of as the sum of the volume of a very large number, n, of pyramids (cones), whose vertices are all at the center of the sphere. And as $n \to \infty$, $h \to r$, $B \to$ a small part of S, and the sums $B \to S$.

$$V_{\text{pyramid}} = \frac{1}{3} Bh$$

Sum of all volumes of pyramids,

$$V_p = \underbrace{\frac{1}{3} h}_{} \underbrace{(B_1 + B_2 + \dots B_n)}_{}$$

As $n \to \infty$,

$$V_{\text{sphere}} = \frac{1}{3} r \cdot 4\pi r^2$$

$$V = \frac{4}{3} \pi r^3 .$$

Lots of intuition needed! [See MJH, Vol. 2, Part II, pp. 533-544].

GRADE 8 - CHAPTER 12

SPATIAL PERCEPTION AND LOCUS

Background:

All of the concepts and figures in this chapter have been encountered before -- points, lines, planes, angles, triangles, circles, spheres, etc.

Purpose:

The chapter is designed

1. to review and summarize the student's previous knowledge about geometric figures and the relationships between them,
2. to extend his intuition and to help him formulate more precisely his insights into the relationships between geometric figures,
3. to introduce locus problems.

Plan:

The material lends itself most easily to class discussion and exploration. For this reason writing a chapter that looks interesting on paper may not be easy. It may be that the best solution is to have short introductory statements followed by sequences of leading questions. The important writing for the chapter would be the Teacher's Commentary. This should contain suggestions for introducing the questions by situations which would catch the student's attention, suggestions for possible ways to guide the discussion, to provoke the appropriate questions, and to use the answers that come from the students to develop the kind of understanding which is being aimed at.

In the first section the aim is to encourage the student to speculate about possible relative positions of a fixed number of points, a fixed number of lines, a fixed number of planes, a given number of lines and planes, etc. In the second section the aim is to let the student establish for himself that there is a smallest number of points which "determine" a line, a plane, 3-space and to find the restrictions which must be placed on them so that the statements hold; similar questions for geometric figures such as triangle, angle, circle, etc. Finally in the third section the aim is to let the student find the set of points which satisfy various geometric conditions. The idea is to pull together and sum up the student's knowledge, experience and intuition about geometric figures in space and their relationships with each other, and to extend these to situations he has not yet met.

12-1. Relationships between two or more given point sets:

1. Two point sets.

   (a) Two points.

   What is the shortest path between two points N and S on the teacher's desk? Is there a different path just as short?

   What is the shortest path between New York and San Francisco? Is there more than one path with this length?

   How good a model of the surface of the earth is a plane? A sphere?

   (b) Two lines.

   Do two lines always lie in a plane? How many planes?

   Suppose two cars are traveling on straight roads, one leading northeast and the other due north. Will there be a junction where the cars can switch roads?

   Do two lines always have a point in common? If they are in the same plane?

If two lines have a point in common, can they have more than one?

(c) Two planes.

Do two planes always meet? If they do meet how many points do they have in common?

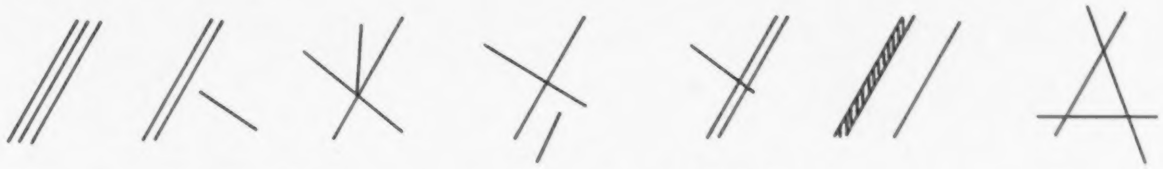
2. Three point sets.

(a) Three points.

Is there a line containing all three points? More than one? Is there a plane containing all three points? How many such planes?

(b) Three lines.

Given three lines, do they always have a point in common? Do at least two of them have a point in common? More than one point in common?



(c) Three planes. (See Int. Math. pp. 433-35)

3. Points and lines.

4. Lines and planes.

12-2. Using a point set to evolve another point set.

1. Points

How many points can you pick arbitrarily if they are to lie on a line?

How many points can you pick arbitrarily if they are to lie in the same plane?

How many points can you pick arbitrarily if they are to lie in 3-space?

Given three points, how many triangles can you form which have these points as vertices? Two of these points as vertices and the other a point on the triangle not a vertex?, etc.

Given three points, can you find a circle which contains these three points on its perimeter? Is there more than one such circle? How many?

Given three points, can you find an angle which contains them? How many such angles? If one point is the vertex of the angle, how many angles contain the three points?

The preceding questions are designed to indicate the ideas which are aimed at in this section. It is hoped that the authors will have imagination and couch the questions in more imaginative ways; e.g., will a three legged stool placed on a smooth floor rock? Why do four-legged tables always seem to rock?

12-3. Sets of points meeting given conditions:

(The aim of this section is to introduce locus problems and to see that the problem is considered in one, two, and three dimensions whenever feasible.)

A treasure is buried on a deserted island. The treasure map says that the treasure chest will be found at the same distance from each of two tall trees 500 yards from the beach. Where should we dig for the treasure?

Concoct other problems which ask for the set of points:

(a) at a fixed distance from a circle.
(b) at a fixed distance from a line segment.
(c) at a fixed distance from a line.
(d) equidistant from a circle.
(e) equidistant from the sides of an angle.
(f) equidistant from the vertices of a triangle, etc.

GRADE 8 - CHAPTER 13

SYSTEMS OF EQUATIONS IN TWO VARIABLES

(See Ch. 15 FCA; Ch. 22 PFCA-h; Ch. 7 Int. Math.)

Background:

Graphs of linear equations.

Solution sets of equations and inequalities.

Graphical solution of systems of equations.

Graphs of simple inequalities, involving strips and half planes.

Purpose:

1. To extend the definition of solutions sets to systems of equations and systems of inequalities.
2. To formulate the concept of equivalent systems and introduce the method of linear combinations for arriving at algebraic solutions.
3. To examine various cases of systems of equations and their graphical interpretation -- inconsistent, consistent, dependent; parallel, coinciding, and intersecting lines.
4. Extend work with systems of inequalities to general linear inequalities and to regions bounded by several straight lines, in preparation for finding convex regions in elementary linear programming problems.

NOTE: We have left locus problems (intuitive geometric notions about point sets in the plane and in 3-space) for a short Chapter 12.

13-1. Solution sets of systems of equations and inequalities:

1. Review definition of wolution set of an equation or inequality.
2. Define solution set for systems of equations and inequalities. Give examples in which the solution set contains no ordered pairs, 1 ordered pair, infinitely many; e.g.,

$$\begin{cases} 2x + y = 5, \\ 2x + y = -2. \end{cases} \qquad \begin{cases} 2x + y = 5, \\ 2x - y = 5. \end{cases} \qquad \begin{cases} 2x + y = 5, \\ 6x + 3y = 15. \end{cases}$$

At this point the solution sets are to be found by examining the graphs.

13-2. Equivalent equations and equivalent systems of equations:

1. Two equations or two systems of equations are equivalent if they have the same solution sets.
2. If an equation in a system of equations is replaced by an equivalent equation the resulting system is equivalent to the original system.
3. Linear combination of left members of two equations (when right member is zero) used to construct simpler equivalent system of equations.
(See Int. Math. Ch. 7, pp. 374-81; also FCA Ch. 15, pp. 468-484)

Example:

$$\begin{cases} 2x - y - 5 = 0, \\ x - 3y + 5 = 0. \end{cases}$$

First replace $2x - y - 5 = 0$ by $a(2x - y - 5) + (x - 3y + 5) = 0$ for an appropriate a. The appropriate one is $a = -3$. The resulting equivalent system is

$$\begin{cases} -5x + 20 = 0, \\ x - 3y + 5 = 0. \end{cases}$$

Now replace $x - 3y + 5 = 0$ by $a(-5x + 20) + b(x - 3y + 5) = 0$. One might choose $a = \frac{1}{5}$ and $b = 1$ or one might take $a = 1$ and $b = 5$.

We eventually get the equivalent system:

$$\begin{cases} x = 4, \\ y = 3; \end{cases}$$

for which the solution set is clearly $\{(4,3)\}$.

13-3. Systems of Linear Equations:

1. Review of graphic solution.
2. Graphical interpretation of linear combination and solution sets.
   Family of lines through a point.
   A look at the possible cases:

L_1 and L_2 the same line $\begin{cases} x + y = 2, \\ 2x + 2y = 4. \end{cases}$

L_1 and L_2 parallel: $\begin{cases} x + y = 2, \\ 2x + 2y = -5. \end{cases}$

L_1 and L_2 intersect in a single points:

$$\begin{cases} x + y = 2, \\ x - y = 4. \end{cases}$$

13-4. Graphical Solution of Systems of Inequalities:

(See FCA pp. 485-492)

Many examples of increasing difficulty and complexity; e.g.,

1. $\begin{cases} y < x, \\ x > 2. \end{cases}$
2. $\begin{cases} |y| < 2, \\ |x| < 1. \end{cases}$
3. $\begin{cases} 2x + 3y < 1, \\ x - y > 2. \end{cases}$
4. $\begin{cases} 2x + 3y < 2, \\ 2x + 3y > 0. \end{cases}$
5. $2x + y > 2,$ $x + y < 1,$ $x \leq 3.$
6. $x + y \leq 5,$ $y \leq 3x + 4,$ $y \leq -x + 4.$

13-5. Applications:

1. Find several word problems which really need two variables to state the conditions. There are some in Chapter 4 on Problem Analysis (see Section 2.9, No. 15 and No. 26, also Section 3.5, No. 9). Devise a number of these which may best be solved algebraically to show the usefulness and power of the methods given earlier in this chapter.

2. Perhaps this is a good time to have a modest discussion of mathematical models as the way in which mathematics is useful in solving problems which arise in the real world.

3. Then a modest introduction to linear programming by means of several examples will use the techniques of Section 13-4 to find the convex region over which we wish to maximize a certain function of two variables.

   (1) The example given in Problem 7 of Appendix C of the Report of the Modeling Committee is a good introduction.

   (2) Diet problems, production problems, etc., will illustrate the usefulness of this idea in many different contemporary business situations.

[For an exposition appropriate for Junior High and as a source of problems at this level, see Chapter 7, pp. 212-222 of Some Lessons in Mathematics, Edited by T. J. Fletcher, Cambridge University Press, 1965.]

CONTENTS OF GRADE 9

Chapter 1: Exponents, Logarithms, Slide Rule

Section 1: Laws of exponents (integral exponents)
Section 2: An exponential function, namely $f : n \to 2^n$
  2.1 Graph of $n \to 2^n$ for $-4 \leq n \leq 5$
  2.2 Extend table and use for computation with numbers as powers of 2
  2.3 Extend laws to include rational exponents
Section 3: Computation Using Powers of 10
  3.1 Construct table of powers of 10, as in Cambridge Report
  3.2 Computations with numbers as powers of 10
Section 4: Introduction of Log Notation
  4.1 Motivate as a simpler notation - re-do table in this form
  4.2 Computation practice
Section 5: Slide Rule Construction and Use
  5.1 A simple slide rule for addition and subtraction
  5.2 Construction of a slide rule for multiplications, using logs
  5.3 Use of a commercial slide rule
Section 6: Exponential and Logarithmic Functions
  6.1 Compare $y = ax$ and $x = \log_a$
  6.2 Graphs of $f : x \to 2^x$ and $L : x \to \log_{2}^{x}$
  6.3 Applications of exponential functions

Chapter 2: Transformations

Section 1: Rigid Motions and Reflections
  2: Projection
  3: Composition of Transformation
  4: Congruence as an Isometric Correspondence
  5: Similarity as a Ratio-Preserving Correspondence
  6: Further Work on Symmetries

Chapter 3
Chapter 4
Systems of Sentences

Nothing available at this time

Chapter 5: Measure Theory

Section 1: Distance as a Function

Section 2: Measure as a Function

- 2.1 Length
- 2.2 Area
- 2.3 Volume

Section 3: Angle Measure as a Function

Section 4: Other Measures

- 4.1 Measurement of a circular arc by means of arc degrees
- 4.2 Area of a curved surface

Chapter 6: Statistics

Section 1:

- 1.1 Organization of data - grouping, histograms
- 1.2 Continuous model of discrete situation
- 1.3 Computation - algorithms for mean, variation, for grouped data

Section 2:

- 2.1 Estimations of mean and variance
- 2.2 Confidence intervals for mean
- 2.3 Chebyshev's Inequality (WST)

Section 3:

- 3.1 Hypothesis testing: (Null hypothesis: quality control, errors of first and second kind)

Section 4

- 4.1 Binomial Theorem
- 4.2 Normal distribution
- 4.3 Central Limit Theorem

Chapter 7: Deductive Reasoning

Section 1: Illustrations of Logical Relationship between Statements
- 1.1 Congruence and similarity for triangles
- 1.2 Similarity of triangles, congruence of two angles of each
- 1.3 Similarity of triangles, congruence of one angle of each
- 1.4 In a triangle, unequal sides and unequal angles
- 1.5 For quadratic function, relation of positive discriminant and real zeros
- 1.6 Medians of triangle, congruence, similarity

Section 2: Suggestions for Geometry topics
- 2.1 Converse of Pythagorean Theorem
- 2.2 Triangle inequality
- 2.3 Standard results on quadrilateral
- 2.4 Circles, chords, secants
- 2.5 Areas of similar triangles proportional to squares of lengths

Section 3: Illustrative Problems
- 3.1 The 30-60 right triangle, the isosceles right triangle
- 3.2 Bisector of angle and division of opposite side of triangle
- 3.3 Failure of initial attempt at angle trisection
- 3.4 Geometric construction of harmonic mean

Chapter 8: Vectors

Nothing available at this time

Chapter 9: Circular Functions

Section 1: Periodic Motion
- 2: Sine and Cosine Functions
- 3: Domain and Range of Sine and Cosine Functions
- 4: The Tangent Function
- 5: Circular Functions and Angles
- 6: Radian Measure

Section 7: Functions of Angles
8: Results of the Definitions (in terms of positive and negative)
9: Numerical Values of Functions in Any Quadrant
10: Graphs of Functions
11: Trigonometry of the Right Triangle
12: An Alternative Suggestion

Chapter 10: Tangency

Section 1: Circles and Line Tangents
- 1.1 The tangent envelope of a circle
- 1.2 Lines tangent to a circle
- 1.3 Constructing tangents
- 1.4 Angles formed by tangents

Section 2: Tangent Lines and Planes in Two and Three Space
Section 3: Circle and Line Tangencies Extended
- 3.1 The relationships of two circles
- 3.2 Common tangent to two circles
- 3.3 Three or more tangent circles

Section 4: Tangent Plane Curves and Tangent Curved Surfaces
Section 5: Tangent Envelopes
- 5.1 The conics
- 5.2 Joining points on a circle
- 5.3 Working in a coordinate system
- 5.4 Pursuit curves

Section 6: Line Tangents to Any Curve
Section 7: Line of Support

Chapter 11: Measure

In progress - not complete at this time

Chapter 12: Complex Numbers

Nothing available at this time

GRADE 9 - CHAPTER 1

EXPONENTS, LOGARITHMS, SLIDE RULE

Background Assumptions:

1. Properties and operations with real numbers.
2. Function concept
3. Some knowledge of positive integral exponents and laws for multiplying and dividing powers of a given base.

Rationale:

The laws of exponents for integral exponents will be developed, and extended to rational exponents.

Computation by using exponents and with the slide rule will be developed as a way of trading multiplication for addition. Development of the idea of computing with powers of 2 will be followed by the construction of a table for powers of 10 which approximate the integers from 1 to 99, following the development in the Cambridge report. Some experience in computation by powers of 10 will lead to the introduction of logarithmic notation. 2^n will be introduced as the function $F : n \rightarrow 2^n$, and the function concept will be used for clarification wherever possible. The log function $L : n \rightarrow \log_2 n$ will be introduced and an intuitive feeling for the functions as inverses of each other will be aimed at, but not formalized.

The table of powers of 10 (i.e., logs of the integers) will be used to construct a "slide rule" on $\frac{1}{10}$ - in. graph paper, from which the manipulation of the slide rule to add and subtract logs will be explained. Having developed this understanding, the student should be prepared to use a commercial slide rule effectively.

Purpose:

1. To review and extend the meaning of exponents and operations with exponents.
2. To introduce the logarithmic function by thinking of it intuitively as the inverse of the exponential function.
3. To develop informally the laws of logarithms.
4. To develop a table of logarithms to base 10 first as powers of 10, and then as logarithms.
5. To provide experiences in computing, using addition of exponents (i.e., logs) to replace multiplication of numbers, etc.
6. To develop understanding of the slide rule and how it operates.

Procedure:

Section 1. Laws of exponents (for integral exponents).

(See First Course in Algebra, Form-H, Chapter 14)

Section 2. An exponential function, $f : n \to 2^n$.

2.1 Construct a table of ordered pairs $(n, 2^n)$ for $-4 \leq n \leq 5$, n integral. Graph the points, and draw a smooth curve. Point out that for $n \to 2^n$ and $m \to 2^m$, $n + m \to 2^n \cdot 2^m = 2^{n+m}$.

2.2 Extend the table in both directions and use it to perform computations such as: $8 \cdot 128 = 2^3 \cdot 2^7 = 2^{10} = 1024$

$$(\tfrac{1}{32})(512) = 2^{-5} \cdot 2^9 = 2^4 = 16$$

$$(\tfrac{1}{4})^3 = (2^{-2})^3 = 2^{-6} = \frac{1}{64}$$

2.3 Justify and develop 2^n for n rational, and extend laws to cover rational exponents. Find points such as $2^{1/2}$ and $2^{3/2}$ on the graph done in 2.1.

Justify and define $2^{2/3} = (2^{1/3})^2 = (2^2)^{1/3}$ as follows:

Let $(2^{1/3})^2 = A$

Then $A^3 = (2^{1/3})^2 \cdot (2^{1/3})^2 \cdot (2^{1/3})^2$

$= [(2^{1/3})(2^{1/3})(2^{1/3})][(2^{1/3})(2^{1/3})(2^{1/3})]$

$= 2 \cdot 2 = 2^2 = 4$

Let $(2^2)^{1/3} = B$

Then $B^3 = (2^2)^{1/3} \cdot (2^2)^{1/3} \cdot (2^2)^{1/3}$

$= 2^2 = 4$

$A^3 = B^3 \rightarrow A = B$

Hence we define $2^{2/3} = (2^{1/3})^2 = (2^2)^{1/3}$.

Note that our law $(2^a)^b = 2^{ab}$ holds for $x > 0$; $x^{r/s} = (x^r)^{1/s}$.

Section 3. Computation using powers of 10.

3.1 Construction of table of powers of 10.

Point out that instead of powers of 2, as in 2.2, powers of 10 could be used for computation, which might be convenient since 10 is the base of our number system. Construct a table of powers of 10, getting the preliminary values as described in the Cambridge report, Appendix B, pp. 73-76.

E.g.,

$$2^{10} = 1024$$

$$2^{10} \sim 10^3$$

$$2 \sim 10^{3/10}$$

(NOTE: In teacher's manual, explain a method of getting corrected values -- see Cambridge report).

It would probably be well to construct as class work powers of 10 corresponding to whole numbers from 1 to 20, and then assign the rest of the whole numbers to 100 by groups, as homework.

3.2 Use the table, just constructed, for simple computations, using scientific notation to get the "characteristic".

Example 1: $160 \times .07$

$$.160 = 16.0 \times 10^{1} = 10^{1.200} \cdot 10^{1}$$
$$0.7 = 7 \times 10^{-1} = 10^{.844} \cdot 10^{-1}$$
$$160 \times .07 = 10^{2.044} = 10^{1.044} \cdot 10^{1} - 11 \cdot 10^{1} = 110$$

Example 2: $262 \div 1.4$

$$262 \sim 26.0 \times 10^{1} = 10^{1.413} \times 10^{1} = 10^{2.413}$$
$$1.4 = 14.0 \times 10^{-1} = 10^{1.144} \times 10^{-1} = 10^{.144}$$
$$\begin{aligned} 262 \div 1.4 = 10^{2.413} \div 10^{.144} &= 10^{2.269} \\ &= 10^{1.269} \cdot 10^{1} \\ &\sim 19 \cdot 10 = 190 \end{aligned}$$

Example 3: $(129)^{4}$

$$129 \sim 13.0 \times 10^{1} = 10^{1.113} \times 10^{1}$$
$$\begin{aligned} (129)^{4} \sim 10^{4.452} \cdot 10^{4} &= 10^{8.452} \\ &= 10^{1.452} \cdot 10^{7} \\ \sim 14 \cdot 10^{7} &= 140{,}000{,}000 \end{aligned}$$

Section 4. Introduction of log notation:

4.1 Since we are expressing positive numbers as approximate powers of 10, it would simplify matters if we had a shorter notation for such a statement as, "The power of 10 which is about equal to 7 is .844".

Explain that there is! We can say "$\log_{10} 7 = .844$" or, more briefly, "$\log 7 = .844$".

Have the class rewrite the table made in 3.1 in this more convenient form, i.e.,

n	log n
1	.000
2	.300
3	.475
	etc.

4.2 Some practice in using this table.

Section 5. Slide Rule Construction and Use:

5.1 Brief consideration of how two number lines can be manipulated to do addition and subtraction. This is trivial in itself, but leads to question: "Could a similar pair of numberlines, with logs of numbers, be manipulated to add logs, thus multiplying the numbers?

5.2 Construct a slide rule, as follows:

(a) Using $\frac{1}{10}$ - inch graph paper and a table of logs, draw a line as in A on the diagram. Line should be 10 inches long, with 10 inches as one unit. Below the line, assign numbers 0, .1, .2, ..., 1 to the appropriate intervals. Above the line, mark off log 1, log 1.1, log 1.2, etc., as shown.

(b) Use ruler to transfer the points marked above line A to line B, drop "lot" from label of each, and write both above and below the line the numerals which indicate the numbers whose logs corresponds to the coordinates of the points on the numberline originally set up in A. This corresponds to the C and D scales in a slide rule.

(c) Line C could be marked as Line A, in 6.1, except that 5" would be the unit. This would give scales corresponding to the A and B scales on the slide rule.

5.3 Use of commercial slide-rule -- show briefly the essentials -- skill will depend upon amount of practicing the student does.

Section 6. Exponential and logarithmic functions:

6.1 Discuss $y = a^x$ and $x = \log_a y$

6.2 Graph $f : x \mapsto 2^x$ and

L+ $x \mapsto \log_2 x$ on same axes.

Point out symmetry with respect to graph of $x \to -x$..

6.3 Applications of exponential functions (See SMSG Elementary Functions, Chapter 4)

Bacterial growth, pp. 146-147
Law of coding, p. 186
Healing of wounds, p. 189
Radioactive decay, pp. 181-182

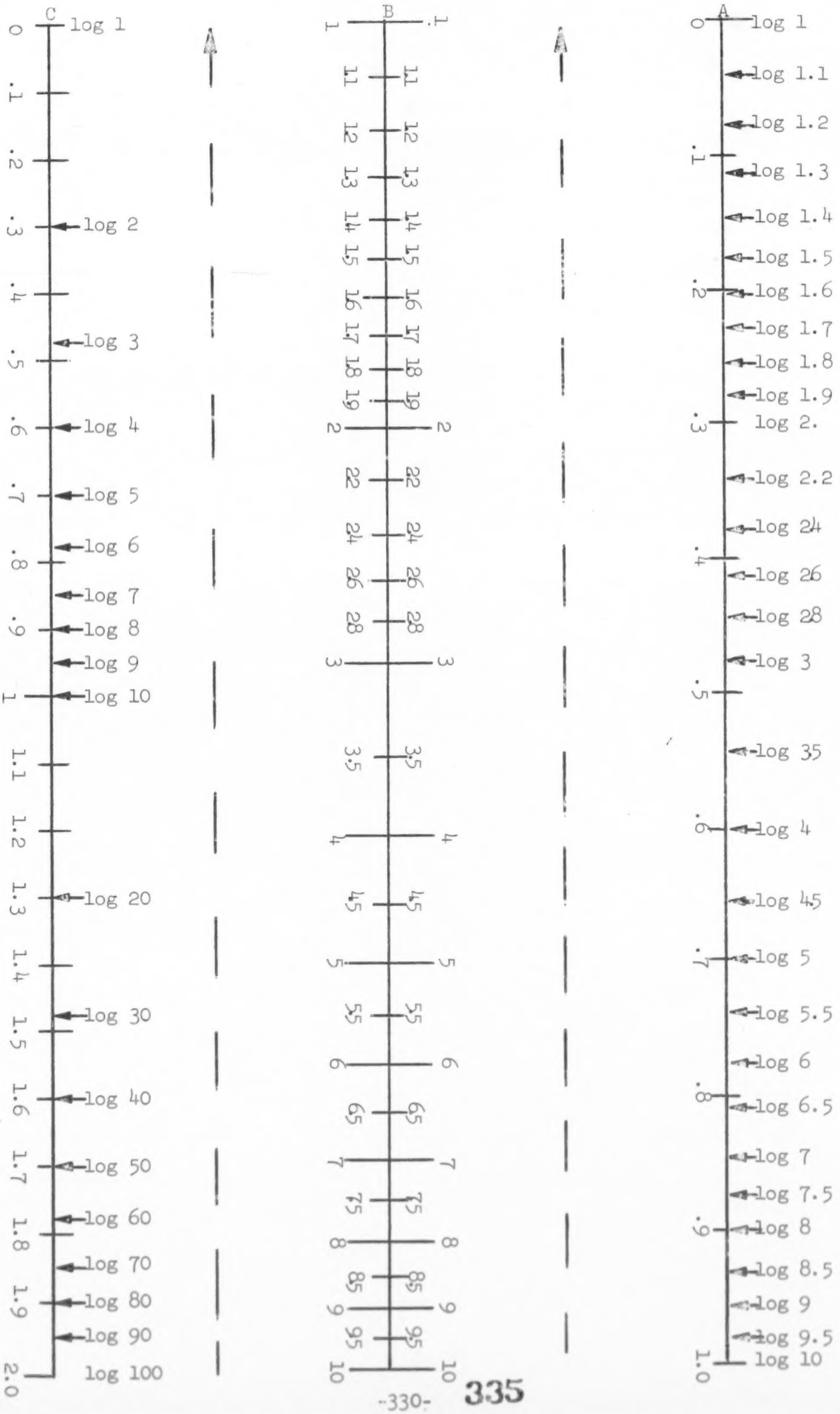

A
log 1
log 1.1
log 1.2
log 1.3
log 1.4
log 1.5
log 1.6
log 1.7
log 1.8
log 1.9
log 2.
log 2.2
log 24
log 26
log 28
log 3
log 35
log 4
log 45
log 5
log 5.5
log 6
log 6.5
log 7
log 7.5
log 8
log 8.5
log 9
log 9.5
log 10
0
.1
.2
.3
.4
.5
.6
.7
.8
.9
1.0
B
1
11
12
13
14
15
16
17
18
19
2
22
24
26
28
3
3.5
4
45
5
55
6
65
7
75
8
85
9
95
10
C
log 1
log 2
log 3
log 4
log 5
log 6
log 7
log 8
log 9
log 10
log 20
log 30
log 40
log 50
log 60
log 70
log 80
log 90
log 100
0
.1
.2
.3
.4
.5
.6
.7
.8
.9
1
1.1
1.2
1.3
1.4
1.5
1.6
1.7
1.8
1.9
2.0

Diagram for Construction of Slide Rule

GRADE 9 - CHAPTER 2

TRANSFORMATIONS

Background:

There are many concepts already introduced which are summarized and used in this chapter. Here is a list of the most important ones:

1. Translations, rotations and reflections
2. Stretches and contractions
3. Vectors; composition of vectors
4. Coordinates in 2 and 3 space
5. Parallel and perpendicular lines and planes
6. Distance, length, congruence
7. Similarity
8. Symmetries of polygons and polyhedra

This shows the great potential of this chapter to integrate many ideas from the past learnings of the students.

Purpose:

The purposes of this chapter are as follows:

1. To combine previous ideas concerning specific transformations and reach the generalization of a rigid motion.
2. To relate certain, special examples of transformations to their form in analytic geometry.
3. To discuss parallel and perpendicular projections of a line on a line, and of a plane on a plane.
4. To consider the projection of a point onto a line or onto a plane as a function; and to extend this to the projection of any geometric figure onto a line or onto a plane.

5. To reconsider stretches and contractions from the newer viewpoint of transformations even though they were introduced earlier in connection with similarity.
6. To present general definitions of both congruence and similarity which will apply to any geometric figures.
7. To advance the ideas of symmetries of polygons along the road toward the concept of groups of transformations.
8. To connect the static idea of symmetry with the dynamic idea of reflection and rotation.

Rationale:

The concept of transformation is basic in mathematics. But, as is true with all such basic ideas, it should be discussed only after students have had much experience with special cases of transformations. By the time this chapter is reached there seems to be enough of this background to summarize and to generalize. So we begin to express the beginning ideas of transformation in a slightly more abstract way. Since there is a great deal in common in the ideas of transformation and function, it is suggested that the writers try to capitalize on these similarities as much as they can.

A second objective is to review and strengthen previous ideas as well as to generalize. Such an objective needs neither explanation nor excuse.

Finally, the possible future extension of the ideas which appear in this chapter is obvious if one considers the analytic definitions of transformations in a plane and in space.

Section 9-1. Rigid motions and reflections:

1.1 Relate transformations to previous idea of vectors; connect translations with new idea of directed line segments.

1.2 Discuss only the following analytic geometry representations of translations:

(a) In 1-space: $f(x) = 0 \rightarrow f(x - h) = 0$

(b) In 2-space: $f(x,y) = 0 \rightarrow f(x - h, y - k) = 0$

(c) In 3-space: $f(x,y,z) = 0 \rightarrow f(x - h, y - k, z - m) = 0$

1.3 Rotations in 2-space about a point; in analytic geometry discuss only the following representations of such rotations:

(a) In polar coordinates: $\rho = f(\theta) \rightarrow \rho = f(\theta - \omega)$

(b) In Cartesian coordinates: Just rotations of 90° and its multiples.

1.4 Rotations in 3-space about lines.

1.5 Reflections in a point; symmetric taken to mean that a figure is its own reflection; in analytic geometry discuss only the following:

(a) In 1-space: $f(x) = 0 \leftrightarrow f(-x) = 0$

(b) In 2-space: $f(x,y) = 0 \leftrightarrow f(-x,-y) = 0$

(c) In 3-space: $f(x,y,z) = 0 \leftrightarrow f(-x,-y,-z) = 0$

1.6 Reflections in a line; symmetric taken to mean that a figure is its own reflection; in analytic geometry discuss only the following:

(a) In 2-space: $f(x,y) = 0 \leftrightarrow f(-x,y) = 0$, etc.

(b) In 3-space: $f(x,y,z) = 0 \leftrightarrow f(-x,-y,z) = 0$, etc.

1.7 Reflections in a plane; symmetric taken to mean that a figure is its own reflection; in analytic geometry discuss the following:

(a) In 3-space: $f(x,y,z) = 0 \leftrightarrow f(-x,y,z) = 0$, etc.

Typical Exercises:

1. Point P has coordinates (2,-3), find the coordinates of P_{90}, P_{180}, P_{270}, P_{360}, which are the results, respectively, of rotating P about the origin of 90°, 180°, 270°, and 360°.

2. A translation moves P(3,4) to P'(5,2). Where does Q(-4,6) go with the same translation?

3. The translation $x' = x + 3$ on a line moves P(5) to P' and Q(7) to Q'. What are the coordinates of P' and Q'? Compute PQ and P'Q' and compare them.

4. The translation $\begin{cases} x' = x + 2 \\ y' = y - 3 \end{cases}$ in 2-space moves P(5,1) to P', and Q(-3,4) to Q'. What are the coordinates of P' and Q'? Compute PQ and P'Q' and compare.

5. How far is each point in space moved by the translation which moves (3,5,6) to (4,7,-2)?

6. In 1-space, which of the following graphs are symmetric in the origin?

   (a) $x + 3 = 0$
   (b) $|x| = 5$
   (c) $x^2 = 16$
   (d) $|x + 3| = 2$
   (e) $x < 3$
   (f) $|x| < 3$
   (g) $|x - 2| > 3$

7. In 2-space, which of the following graphs are symmetric in the origin? Which in the x-axis? Which in the y-axis?

   (a) $y = x^2$
   (b) $xy = 12$
   (c) $x^2 + y^2 = 4$
   (d) $x + y = 5$

8. In 3-space, is the graph of $x^2 + y^2 + z^2 = 16$ symmetric in the origin? In what lines and planes is it symmetric?

Section 2. Projection:

2.1 Parallel projections of space; equivalent to translations and vectors.

2.2 In 2-space, parallel projections from line to line; use both parallel lines and intersecting lines.

2.3 In 3-space, parallel projections from plane to plane; use both parallel planes and intersecting planes.

2.4 Central projections of space; equivalent to stretches and contractions.

2.5 In 2-space, central projection from line to line; use both parallel lines and intersecting lines.

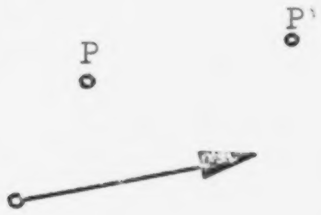
2.6 In 3-space, central projection from plane to plane; use both parallel planes and intersecting planes.

2.7 In 2-space, perpendicular projection of point on line as function; the projection on a line of a point, a segment, a geometric figure.

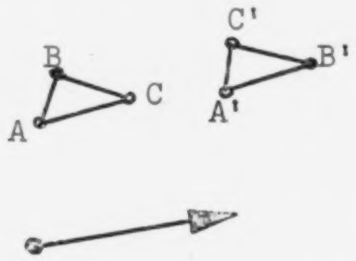
2.8 In 3-space, perpendicular projection of point on plane as function; the projection on a plane of a point, a segment, a region, a geometric figure.

The following diagrams will be helpful in connection with the foregoing topics. Each diagram has the topic numbers beside it to which it pertains.

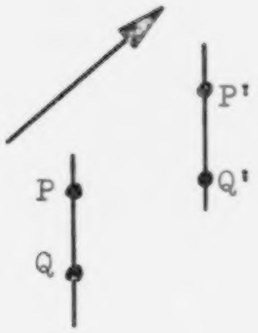
2.1



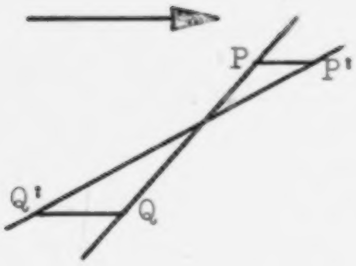
2.1



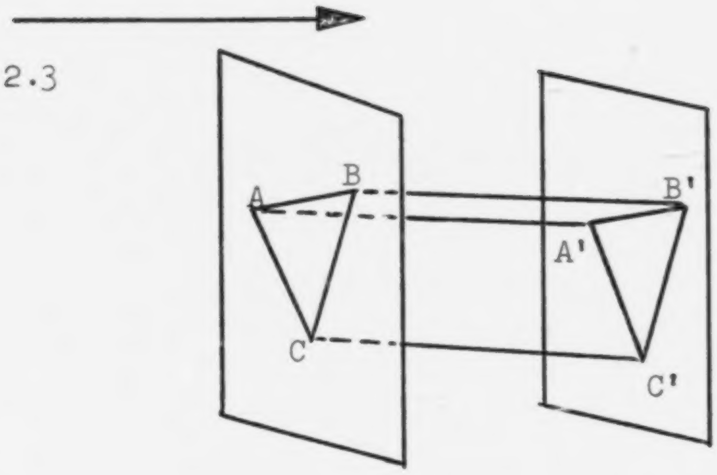
2.2



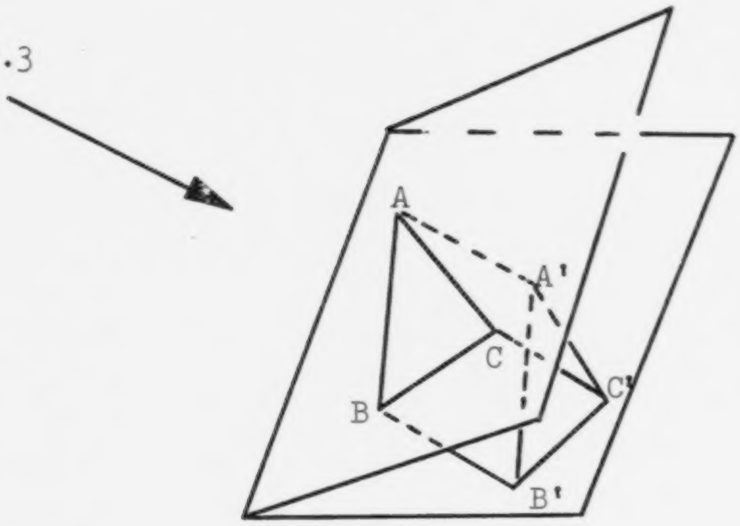
2.2



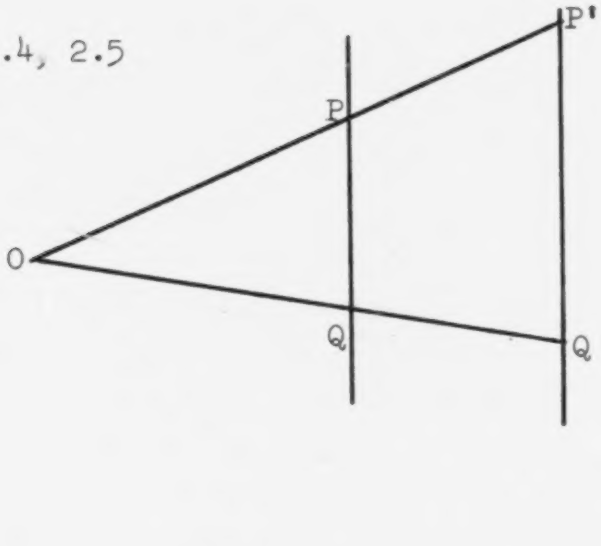
2.3



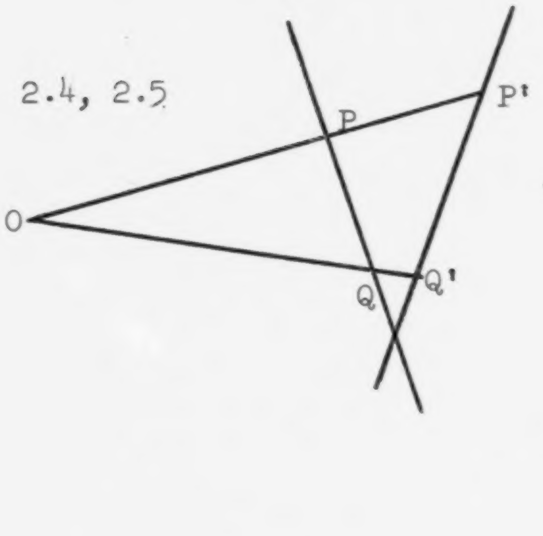
2.3



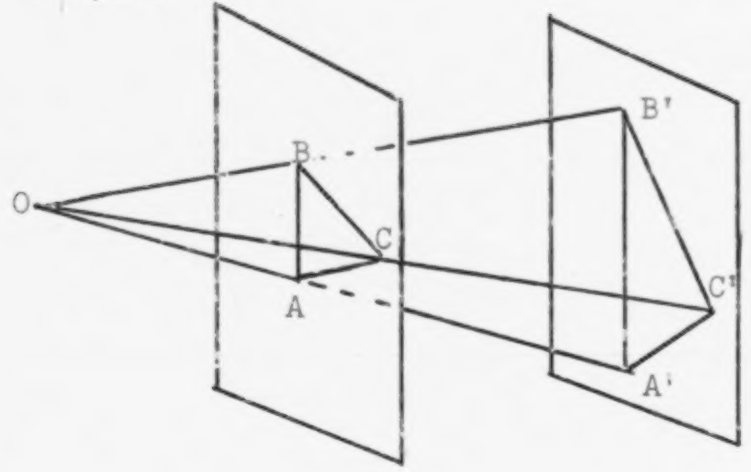
2.4, 2.5



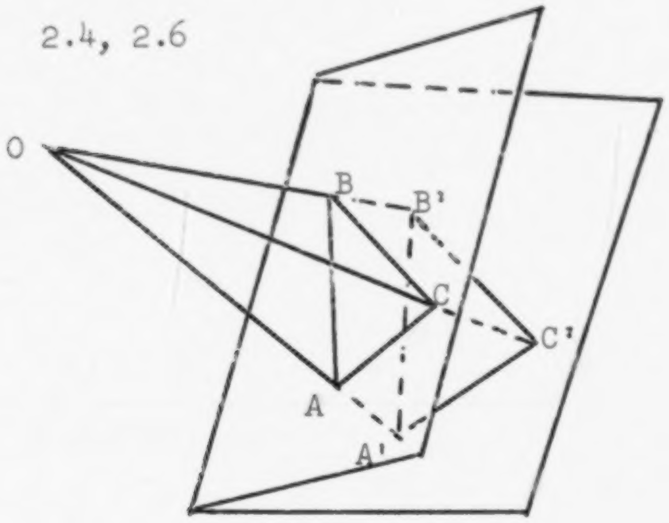
2.4, 2.5



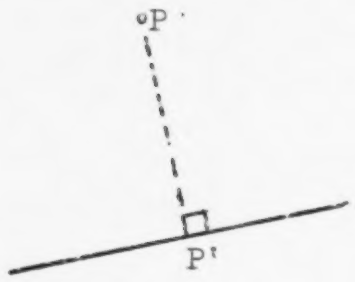
2.4, 2.6



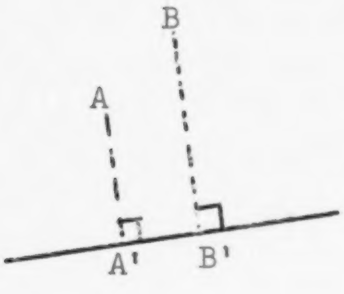
2.4, 2.6



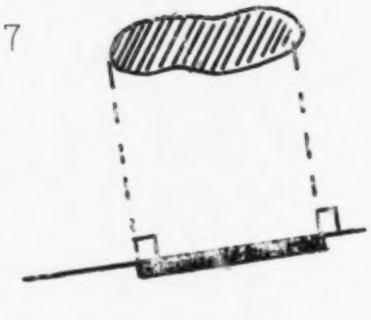
2.7



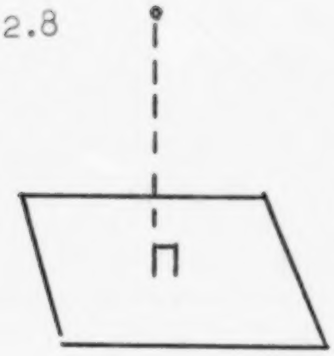
2.7



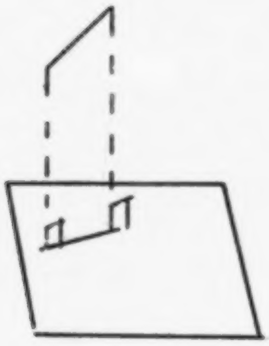
2.7



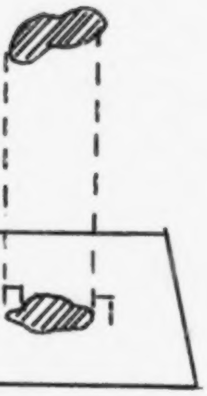
2.8



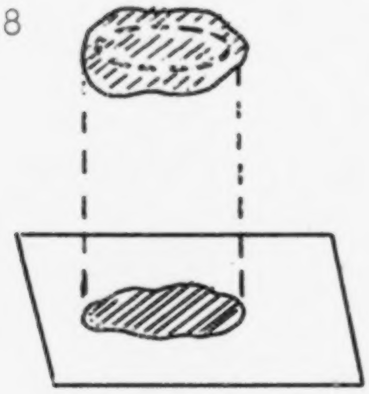
2.8



2.8

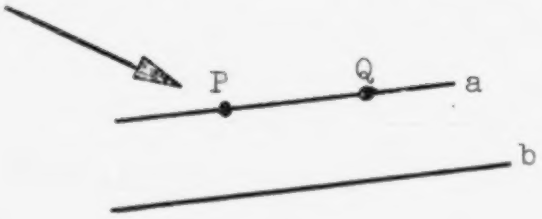


2.8

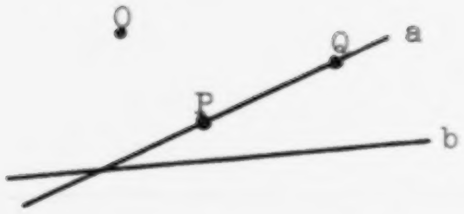


Typical Exercises:

1. Consider the projection parallel to the direction d. Copy the figure and mark P' and Q', the projections of P and Q, respectively, from line a to line b.



2. a and b are intersecting lines. Copy this figure and mark P' and Q' the central projection (with center O) from a to b of P and Q.



3. Is the projection of a triangle onto a plane always a triangle?

4. A segment has length 5". What can you say about the length of its projection onto a given plane?

Section 3. Composition of transformations:

3.1 Composition of two translations.

3.2 Composition of two rotations about point in 2-space; or about line in 3-space.

3.3 Composition of 2 reflections in 2-space in two parallel lines; in 2-space in 2 intersecting lines; in 3-space in 2 parallel planes; in 3-space in 2 intersecting planes.

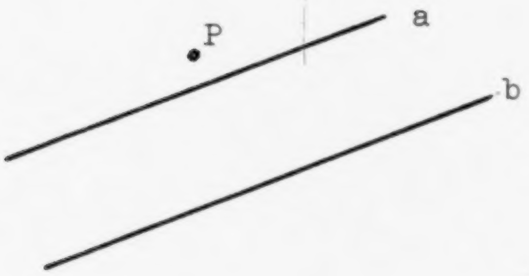
3.4 Rigid motions as composition of translations and rotations; composition of rigid motions.

3.5 Composition of stretches and contractions with rigid motions and reflections.

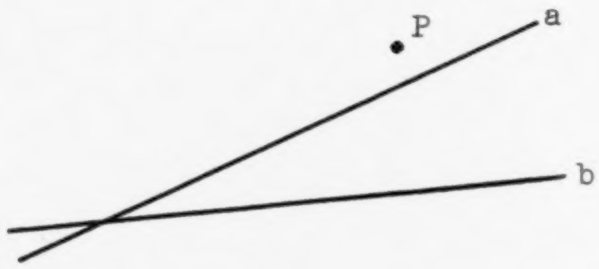
Typical Exercises:

1. Point P is first reflected in line a to get P'; then P' is reflected in line b to get P". Mark P' and P" for each of these situations:

(a)

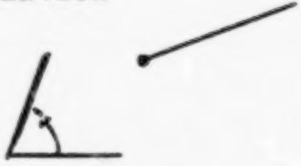


(b)

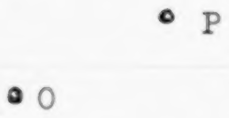


2. Consider the translation

and the rotation



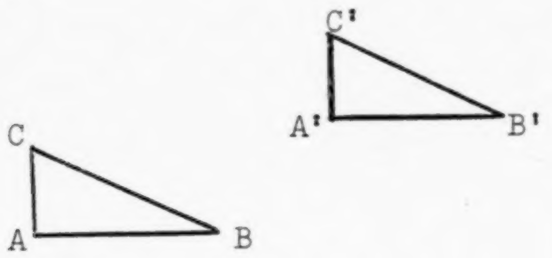
about O. P is carried to P' by the translation, then P' is carried to P" by the rotation. Copy the figure and mark the positions of P' and P".



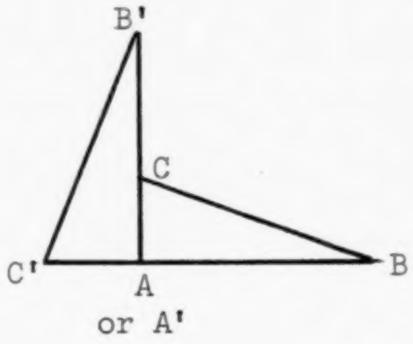
3. On the same diagram as 2 first carry P to Q' by the rotation, and then carry Q' to Q" by the translation. Is P" the same as Q"?

4. Invent a combination of transformations that will carry triangle ABC into triangle A'B'C' for each of the following cases:

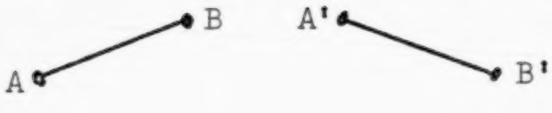
(a)



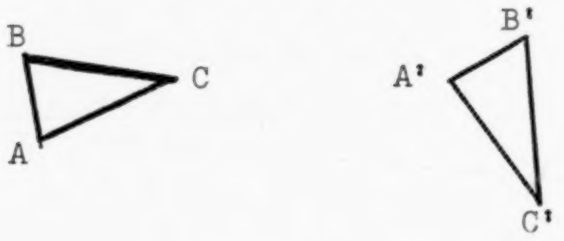
(c)



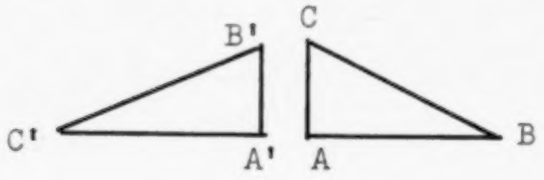
(b)



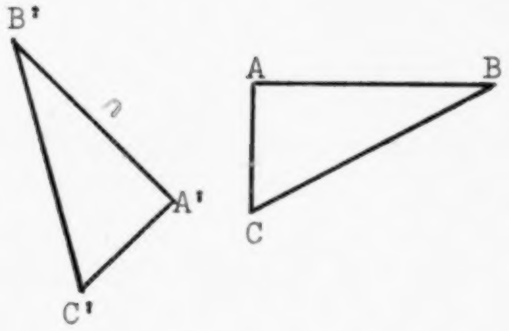
(d)



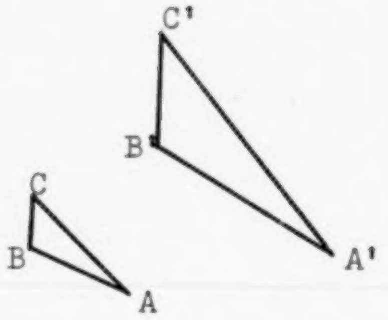
(e)



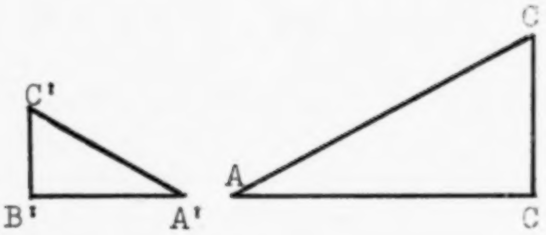
(f)



(g)



(h)

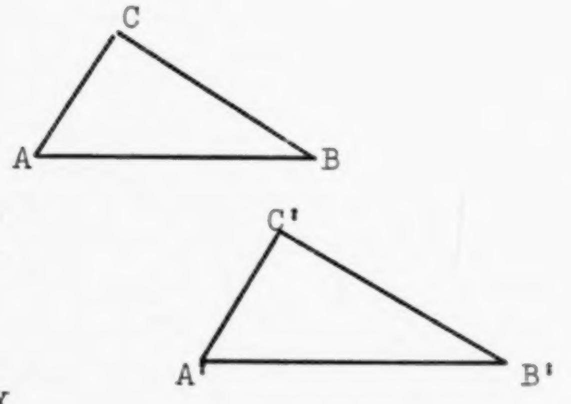


Section 4. Congruence as an isometric correspondence:

4.1 Show that congruence can be thought of as a distance preserving correspondence.

Typical Exercise:

1. We are given triangle ABC and A'B'C' with this correspondence of points set up:



X	f(X)
A	A'
B	B'
C	C'
X $\in \overline{AB}$	X' $\in \overline{A'B'}$ so that A'X' = AX
X $\in \overline{BC}$	X' $\in \overline{B'C'}$ so that B'X' = BX
X $\in \overline{AC}$	X' $\in \overline{A'C'}$ so that C'X' = CX

Prove that $\triangle ABC \cong \triangle A'B'C'$ by proving that if X and Y are points of $\triangle ABC$, then X'Y' = XY for all X and Y.

Section 5. Similarity as a ratio preserving correspondence:

5.1 Show that similarity can be thought of as a ratio preserving correspondence.

5.2 Demonstrate similarity of odd shaped 2-dimensional and 3-dimensional figures.

5.3 Discuss similarity as composition of congruences and stretches and contractions.

5.4 Discuss congruence as special case of similarity, or similarity as a generalization of congruence.

5.5 Slicing similar polygons into similar triangles.

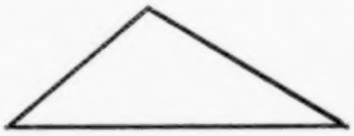
5.6 Similar tetrahedrons, pyramids and primes; all cubes are similar; all regular tetrahedrons are similar.

5.7 Similar cylinders, cones and spheres; all spheres are similar

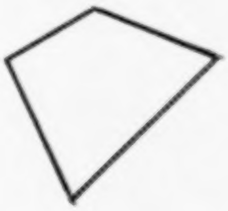
Typical Exercises:

1. Find three ways of drawing a figure similar to each of these, but twice the size in each linear dimension.

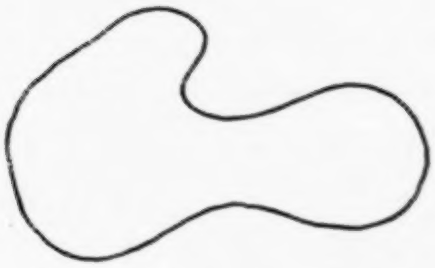
(a)



(c)

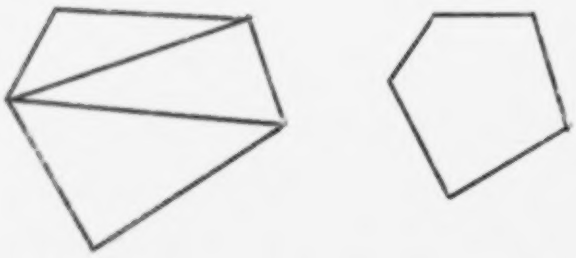


(b)

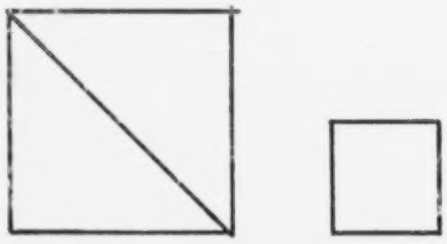


2. We wish to slice a polygon into triangles so that the vertices of the triangles are a subset of the vertices of the polygon. In each of the following exercises one figure has been out into such triangles. Discover how many different ways the other, similar figure can be cut into triangles so that they are respectively similar to the triangles in the first figure.

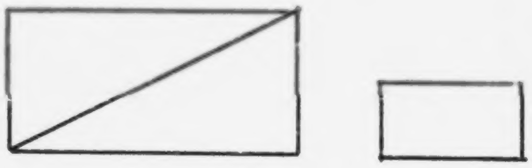
(a)



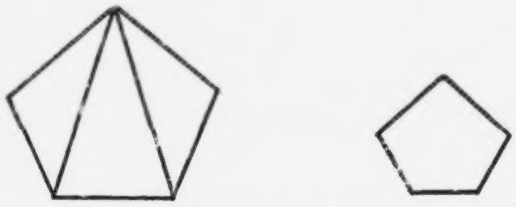
(c)



(b)



(d)



regular pentagons

Section 6. Further work on symmetries:

6.1 Review symmetries of isosceles triangle, non-square rectangle, equilateral triangle and square (See Grade 8, Chapter 10, "Parallels and Perpendiculars", Section 5.4 through 5.7).

6.2 Make tables of compositions of transformations of these figures into themselves. Note that the order above is in the order of increasing difficulty. Make clear whether rules allow flips to the other side or not; first do without flips and then with them.

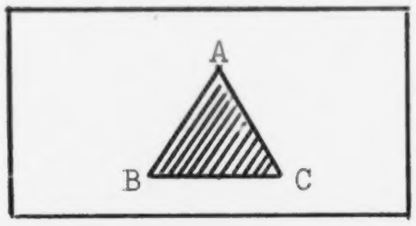
6.3 Consider properties common to the tables of 6.2 in order to discuss the properties of a group; relate to other groups: modular number systems, operations with numbers.

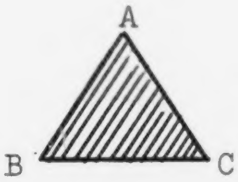
6.4 Transformations of a cube into itself, of a tetrahedron into itself; probably not continued to the composition of the transformations.

Typical Exercise:

The usual approach to symmetries is by way of motions (reflections, spins, flips) which "leave the figure unchanged". It is now suggested that a new approach to the same problem be made by a slightly more abstract method. We will remove the need for motion. The following exercise shows what this means.

1. You have two objects: a board with a hole in it, and a block of wood which will fit into the hole. First let us consider that both the hole and the block are equilateral triangles:





On both the block and around the hole are painted letters to identify the vertices. The letters appear on both sides of the block. In how many ways can the block be placed in the hole? List these ways.

Note: The answer might begin like this:

Identification of way	Next to vertex		
	A	B	C
1	A	B	C
2	A	C	B
3	B	A	C

etc.

Then a composition of "substitutions" might lead to a table of operations which could start like this:

	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	1				
3	3					
4	4					
5	5					
6	6		etc.			

The purpose of the exercise is to point in the direction of permutation groups. It is interesting to students that all permutations of ABC appear in this equilateral triangle problem, but not all permutations of ABCD are used when we do the corresponding problem for a square.

GRADE 9 - CHAPTER 5

MEASURE THEORY

Condensed Outline:

Measure theory (descriptive and semiformal).

Subdivisibility, additivity, invariance under congruence.

Application to length, angle, area, volume.

Rationale:

Chapter 5 serves as an interlude in the development of an appreciation for numbers as used in geometry. The student has been exposed to virtually all the traditional formulas for mensuration of geometrical figures, so many that he may have lost a perspective on the basic principles. In general each of these formulas has been accorded a justification appropriate to the student's level when it was introduced. Now is an opportunity to pause and survey the accomplishments spread over several years, examining from a considerably more mature viewpoint the fundamental ideas without paying attention to the multitude of detailed consequences of the basic principles. Customarily a chapter listing the "postulates" for this portion of geometry includes a long list of derived results that may permit the student to feel security in the details without grasping fully the principles; we have selected to build an entire chapter on the fundamentals so that their importance will not be underestimated.

What to accomplish?

What are the fundamental properties of measure?

Sometimes we measure how far apart two figures are, sometimes how big a figure is.

(a) In the first category, we talk about how far apart for two points, two parallel lines, a line and a point, two parallel planes, etc. All these are distances. Review that the basic idea is the distance between a pair of points, since the other applications are obtained from it together with the notion of a minimum. What is distance? How is distance related to other aspects of geometry?

(b) In the second category, we use different words (length, area, volume) in describing how big a figure is, depending on whether the figure is one-dimensional, two-dimensional, or three-dimensional. All of these can be collected under the heading of "measure". In the one-dimensional case we begin by discussing the length of a "straight" figure, a segment. We extend to figures that are not straight, but are composed of straight pieces (perimeter of polygon, or length of polygonal path) and later we extend to length of curve. In the two-dimensional case we begin with the area of a very simple figure, a rectangular region, a region with a simple type of boundary. We extend to other regions whose boundaries are composed of straight pieces and eventually are able to consider regions enclosed by curves. An analogous discussion applies in three dimensions. In any of these cases, with what do we begin as fundamental? How do we do the extensions? Just what is a measure? How is it related to other aspects of geometry? Is there a relation between measure and distance?

(c) A third category, not mentioned above, concerns angles. From one viewpoint, the measure of an angle may be thought of as telling how big the angular region is, and thus belongs to category (b). But from another viewpoint the measure of an angle may be thought of as telling how far apart the sides of the angle are, how far apart in the sense of rotation, and thus belongs to category (a). So angle measurement has a somewhat special role. Just what properties does it have of one type, and what of the other type?

After posing the above questions, we try to agree on some answers.

1. Distance

First, distance is a function. Each (unordered) pair of points in space belongs to the domain; the range is a set of nonnegative numbers.

Second, the image of a pair of distinct points is a positive number. (The image of each pair of coincident points is zero.)

Third, if A, B, C are any points, then $d(A,C)$ is equal to or is less than $d(A,B) + d(B,C)$ according as B is or is not a point that is (collinear with and) between A and C.

The first two properties emphasize what distance is, while the third relates distance to other aspects of geometry.

2. Measure

(a) Length

First, measure is a function. Its range is a set of nonnegative numbers. Its domain consists of various one-dimensional sets; among these are all segments, all polygons, all circles, and (intuitively) various other sets that WST we do not identify fully.

Second, the measure of a segment is the same number as the distance between the endpoints.

Third, a set congruent to a measureable set is also measurable and the measures are the same.

Fourth, the union of finitely many measurable sets, no two of which "overlap", is measurable and its measure is the sum of the measures of the individual sets.

Tread extremely lightly on the possibility of a set without measure, but leave the door open.

The first property emphasizes what length is. The second identifies that the starting place for a development is the measure of a segment and also ties together the notions of measure and distance. The third property relates measure to the congruence idea. Whereas the second property associates distance with measure for segments, the fourth property permits extension of this association by considering the measure of a polygonal path (or polygon).

We mention a fifth property that helps us in describing the concept of length of a curve. Thus far you have had only one or two opportunities to apply this property, although it will be used extensively in your later mathematics. We shall not formulate it as carefully as we do the others because we do not have enough mathematical background.

Fifth, if a measurable set is a good enough approximation to another measurable set, then the measures are approximately the same.

Our difficulty in formulating this property is the vagueness about an approximation being "good enough".

(b) Area

First, area is a function. Its range is a set of nonnegative numbers. Its domain consists of various two-dimensional sets; among these are all rectangular regions (that is, two-dimensional intervals), all convex (bounded) polygonal regions, all circular discs, and (intuitive) various other sets that WST we do not identify fully.

Second, the measure of a rectangular region is the product of the measures of segments forming two sides of its boundary with a common endpoint.

Third, a set congruent to a measurable set is also measurable and the measures are the same.

Fourth, the union of finitely many measurable sets, no two of which "overlap", is measurable and its measure is the sum of the measures of the individual sets.

Fifth, if a measurable set is a good enough approximation to another measurable set, then the measures are approximately the same.

The first property emphasizes what area is. The second identifies that the starting place for a development is the measure of an interval and also ties together the notions of area with length and hence with distance. The third, fourth, fifth properties are copies of the corresponding ones for length.

(c) Volume

First, volume is a function. Its range is a set of nonnegative numbers. Its domain consists of various three-dimensional sets; among these are all rectangular parallelepipedal regions (that is, three-dimensional intervals), all convex (bounded) polyhedral regions, all spherical balls, and (intuitively) various other sets that WST we do not identify fully.

Second, the measure of an interval is the product of the measures of segments forming three sides of its boundary with a common endpoint.

Third, a set congruent to a measurable set is also measurable and the measures are the same.

Fourth, the union of finitely many measurable sets, no two of which "overlap", is measurable and its measure is the sum of the measures of the individual sets.

Fifth, if a measurable set is a good enough approximation to another measurable set, then the measures are approximately the same.

The comments on these properties are the analogs of the remarks on area. The fact that most of the features of measure are the same for all dimensionalities should now be clear.

3. Angle measure

First, measure is a function. Its domain is the set of all angles. Its range consists of real numbers between 0 and c, for some appropriately chosen positive number c. In degree measure $c = 180$, while in other schemes c may be another positive number.

Second, two congruent angles have the same measure.

Third, if B lies in the interior of $\angle AVC$, then $m\,\angle AVB + m\,\angle BVC = m\,\angle AVC$.

4. Other measures

   (a) An example of a derived measure is the measurement of a circular arc by means of arc degrees, where this assignment is based on the measure of the central angle and the issue of whether the arc is a major arc or a minor arc.

   (b) Another measure is illustrated by the area of a curved surface, a topic we prefer to throw open for theoretical consideration to the best students only.

The entire body of material in Rationale should be brought into focus. Many applications of the ideas have been used during the past few years on the student's training on some sort of basis. The more important of these should be reviewed in order to assist in formulating the principles. (Examples include the area of a triangular region as half the area of the interior of a corresponding parallelogram, or the volume enclosed by a tetrahedron as one-sixth the volume of the interior of a corresponding prism, or the length of a circle as being approximated by the length of a polygon, or every regular hexagon with side of a given length as the same area for the enclosed region.)

After the principles have been formulated and have been further illustrated by known examples, further knowledge should be stimulated about such questions as:

(a) the length of a curve such as the boundary of a normal window,
(b) the shortest polygonal path that passes through specified points,
(c) the perimeter of spherical triangle (of simple type),
(d) the area of the interior of an ellipse -- exploratory and intuitive!,
(e) the volume of a doughnut -- exploratory and intuitive!

The long-range forward look is toward the measure ideas treated in the calculus. Tie-in strongly with probability notions also!

GRADE 9 - CHAPTER 7

DEDUCTIVE REASONING

A. This chapter in Grade 9 was formerly listed by the Geometry Committee under the title "Triangles: Deductive Treatment".

B. Purpose (in brief): To give an elementary approach to deductive reasoning, in preparation for the more serious level of rigor at Grade 10.

C. Goals (in more detail):

(1) An introduction to the notions of axiom, definition, theorem, proof.
(2) The nature of "if ..., then ...".
(3) Distinction between a conditional statement and its converse.
(4) Practice with contrapositives (WST: enough practice so that the very top-notch student may discover for himself that a contrapositive is logically equivalent to the original).
(5) Proofs by elimination of all but one alternative (as one type of "indirect" proof).
(6) Proof by one example: often used to accomplish a disproof by counterexample.

D. Subject matter (in summary): Select material from algebra and from the synthetic study of geometry, especially designed to meet the above goals. Although part of the material may be familiar to the student from an informal approach, a significant portion should be new.

E. A few illustrations of the logical relationship between statements:

(1) If two triangles are congruent, then they are similar. If two triangles are similar, then they are congruent. If two triangles are not similar, then they are not congruent.

(2) If two triangles are similar, then two angles of one triangle are congruent respectively to two angles of the other triangle. If two angles of one triangle are congruent respectively to two angles of another triangle, then there is a similarity of the one triangle onto the other.

(3) If two triangles are similar, then one angle of one triangle and one angle of the other triangle are congruent to each other. If one angle of one triangle and one angle of another triangle are congruent, then the triangles are similar.

(4) Accepting that "In a triangle the angle opposite the longer of two sides has a greater measure than the angle opposite the shorter side", deduce that "In a triangle the side opposite the larger of two angles is longer than the side opposite the smaller angle".

(5) If a quadratic function has a positive discriminant, then the function has two (real) zeros. If a quadratic function has two (real) zeros, then it has a positive discriminant. If a function has two (real) zeros, then it is a quadratic function with a positive discriminant.

(6) Two medians of one triangle have lengths 20 and 12; two medians of another triangle have lengths 16 and 10. Can the two triangles be congruent? If so, how? Can they be similar? If so, how? By more than one correspondence?

F. Some suggestions for geometry topics to be developed:

(1) The converse of the Pythagorean theorem.

(2) The triangle inequality ($AB + BC > AC$ if A, B, C are noncollinear). (Remark: Preparation for chapter on Vectors later in Grade 9.)

(3) A treatment of standard results on quadrilaterals, extending the student's knowledge attained in earlier grades.

(4) A deductive chain of theorems concerning circles, their chords and secants. (Remark: A later chapter in Grade 9 is entitled Tangency.)

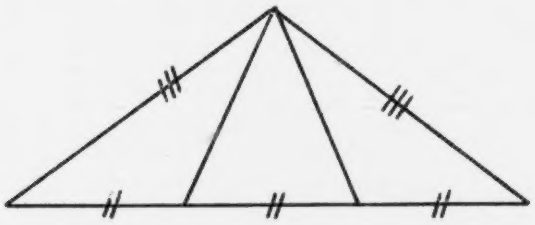
(5) Areas of similar triangles are proportional to the second powers of the lengths of any two corresponding sides or any two corresponding altitudes. (Remark: This is another step on the spiral to a more detailed study of areas and lengths under similarity that is scheduled for a later chapter in Grade 9.)

G. A few illustrative problems (perhaps illustrating the upper bound on difficulty):

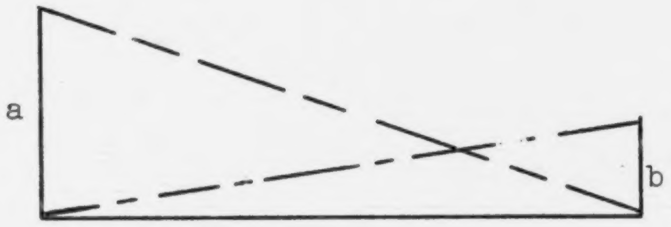
(1) Concerning right triangles:
30-60-90 if and only if hypotenuse is twice as long as one leg; isosceles if and only if hypotenuse is $\sqrt{2}$ times as long as one leg.

(2) In a triangle the bisector of an angle separates the opposite side into segments that are proportional to the adjacent sides of the triangle.

(3) Prove the failure of the initial attempt at angle trisection.



(4) Geometric construction of the harmonic mean (accompanied by real world applications of the harmonic mean, not called by this name.



GRADE 9 - CHAPTER 8

VECTORS

Outline Only

1. Abstract from displacement concept
2. Operations on vectors
3. Decomposition
4. Association (intuitive) of vectors with analytic geometry
5. Vector geometric proofs (simple)
6. Length of vectors

GRADE 9 - CHAPTER 9

CIRCULAR FUNCTIONS

Background Assumed:

1. Basic geometric concepts - angle, degree, properties of the right triangle, relation of the measure of an arc to the measure of its corresponding central angle, properties of the isosceles right triangle and the 30-60-90 degree triangle.
2. Coordinate systems in two dimensions.
3. Ability to work with radicals.
4. Ratio and proportion.

Purpose:

1. To introduce general definitions of the sine, cosine and tangent functions.
2. To discuss some of the basic properties of these functions.
3. To introduce radian measure.
4. To derive numerical values of the above functions for the quadrantal angles and angles such as 30°, 135°, 300°, etc.
5. To define these functions for the coordinate free right triangle.
6. To solve simple verbal problems involving the right triangle.

Procedure:

1. Periodic motion and its prevalence.
   - (a) succession of day and night
   - (b) change of seasons
   - (c) passage of second hand on a watch over a specified point

(d) spring

(e) vibrating string

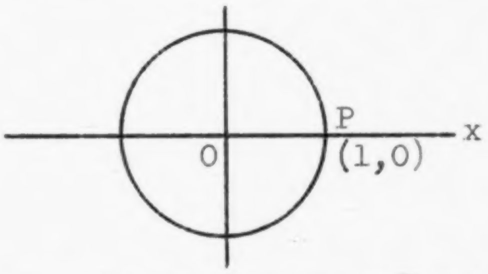
(f) cams (circular)

(g) pistons (circular)

Point out that all periodic motion has circular motion as a model, of course, not necessarily uniform motion.

2. Sine and cosine functions.

Consider the unit circle with circumference 2π. If we study the motion of a point P as it moves along the circle in a counter-clockwise direction, we can locate P exactly by knowing how far it has traveled along the circle from the point (1,0).



The distance it has covered is the length of the arc from its starting point to its stopping point. We shall regard motion in a counter-clockwise direction as positive and motion in a clockwise direction as negative.

At every point in its progress, point P is associated with an ordered pair of real numbers. We may say that the motion of P defines a function f. With each arc length we associate an ordered pair of real numbers (x,y), the coordinates of p, or

$$f : a \rightarrow (x,y)$$

where a is the distance traveled. Since it is inconvenient to work with a function whose range is a set of ordered pairs and since each coordinate is itself a function of a we define two functions, as follows:

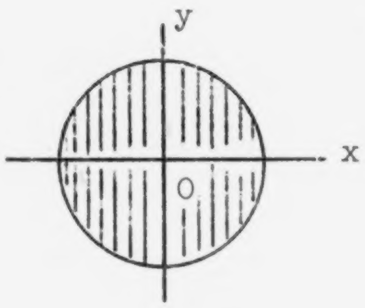
Cosine: $a \rightarrow x$, where x is the ordinate of the point determined by the distance.

Sine: $a \to y$, where y is the abscissa of the point determined by the distance.

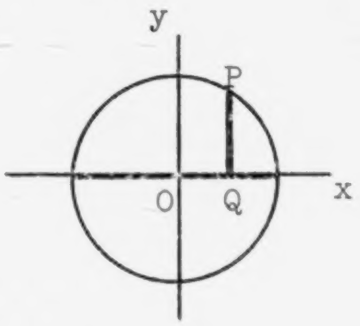
From the definition of the sine function it follows that this function is periodic with a period of 2π. For every arc length of 2π the correspondence of a and x is repeated.

A similar comment may be made with reference to the cosine function.

3. Domain and range of sine and cosine functions.



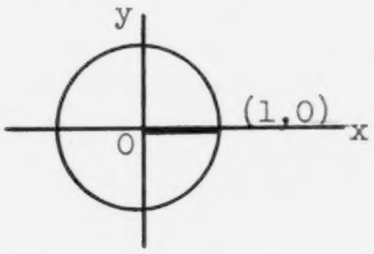
The sine function maps the arc length into the ordinate. The diagram shows this correspondence. The domain of the function is the set of real numbers since the arc length may represent any number of revolutions either in a positive or a negative direction. The range of the function is $-1 \leq \sin a \leq 1$ since the ordinate never exceeds 1, or falls below -1 (see diagram).



The cosine function maps the arc length into the abscissa. The diagram indicates this correspondence. The domain of the function is the set of real numbers since the arc length may represent any number of revolutions either in a positive or a negative direction. The range of the function is $-1 \leq \cos a \leq 1$ since the abscissa never exceeds 1, or falls below -1 (see diagram).

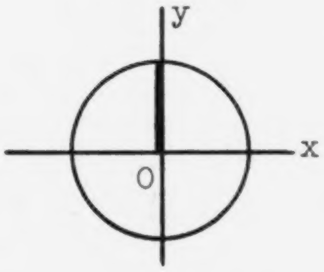
We now have the following results:

$\cos 0 = 1$



$\cos 0$ - corresponding to the arc length 0 we have the abscissa 1.

$\cos \frac{\pi}{2} = 0$



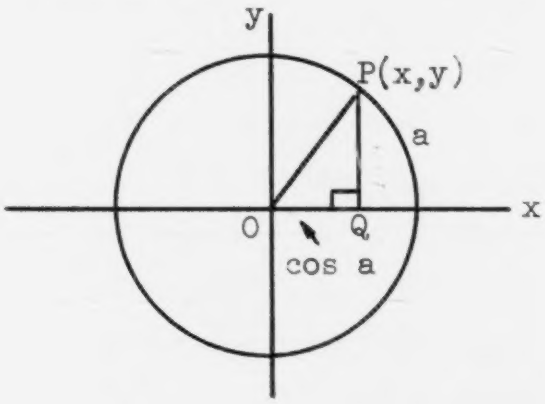
$\cos \frac{\pi}{2}$ - corresponding to the arc length $\frac{\pi}{2}$ we have the abscissa 0.

In a similar manner, we show diagrammatically that

$$\cos \pi = -1, \quad \cos \frac{3\pi}{2} = 0, \quad \cos 2\pi = 1$$

$$\sin 0 = 0, \quad \sin \frac{\pi}{2} = 1, \quad \sin \pi = 0, \quad \sin \frac{3\pi}{2} = -1, \quad \sin 2\pi = 0.$$

There is an alternative method of determining the domain and range of the sine and cosine functions as follows:



For the arc length AP, or a,

$$\text{sine } a = PQ, \quad \text{cosine } a = OQ.$$

In the right triangle OPQ,

$$(OP)^2 = (OQ)^2 + (PQ)^2$$

$$1 = \cos^2 a + \sin^2 a$$

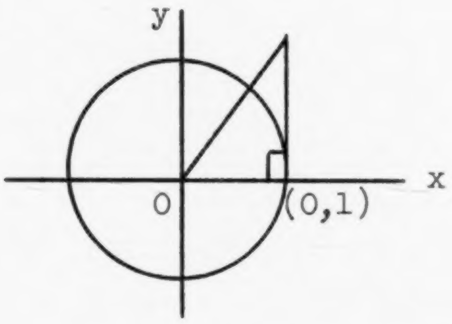
An examination of this equation reveals that $\sin a \leq |1|$ and $\cos a \leq |1|$.

4. The tangent function.

Introduce the tangent function as follows:

tangent: $a \rightarrow \frac{x}{y}$, where x is the abscissa of the point determined by the distance, a, and y is the ordinate of the point determined by the distance, a.

As the diagram indicates, the tangent function is undefined when the arc length is $\frac{\pi}{2}$ or $\frac{3\pi}{2}$. When the arc length is $\frac{\pi}{2}$ or $\frac{3\pi}{2}$ the ordinate is 0.



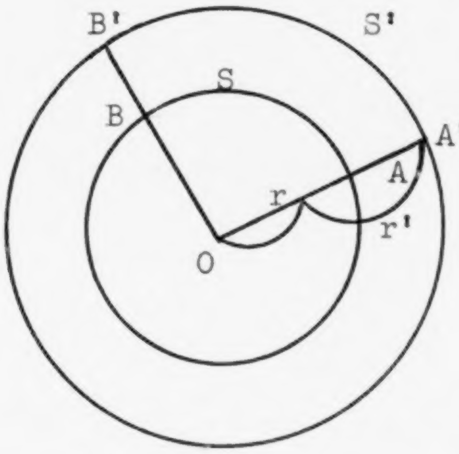
5. Circular functions and angles.

The circular functions are closely related to the functions of angles. In establishing degree measure, we can divide the circumference of the circle of unit radius into 360 equal arc lengths and measure a central angle by the number of units of arc length it includes. For example, if an angle includes $\frac{1}{4}$ of the circumference, or $\frac{\pi}{2}$ units of length, we say that the measure of the angle is $\frac{1}{4} \times 360^\circ$, or 90°. Thus $\sin \frac{\pi}{2} = \sin 90^\circ = 1$. We will soon define the sine, cosine, and tangent functions of angles.

6. Radian measure.

We introduce another unit for measuring angles, the radian.

In unequal circles, arcs subtending equal angles have the same ratio as the corresponding radii. In the figure below



$$\frac{S}{S'} = \frac{\overline{OA}}{\overline{OA'}} = \frac{r}{r'} \quad \text{or} \quad \frac{S}{r} = \frac{S'}{r'}$$

If $S = r$, the ratio $\frac{S}{r} = \frac{S'}{r'} = 1$.

If we select as a unit of measure of the central angle an arc whose length is equal to the radius, we have a new unit of measure called the radian. This unit of angle measure is independent of the length of the radius of the circle.

Since $C = 2\pi r$, an arc whose length is r can be laid off exactly 2π times to complete one rotation. Thus, one complete rotation requires 2π radians in radian measure, or 360° in degree measure. Hence,

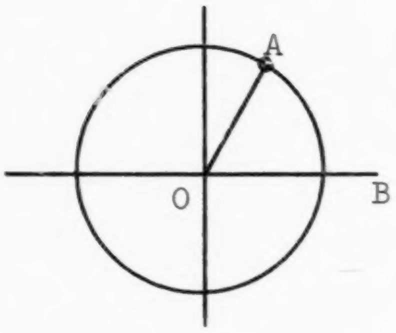
$$2\pi \text{ radians} = 360^\circ.$$

The following proportion may be used in converting from degrees to radians, and vice versa:

$$\frac{\text{number of degrees}}{360} = \frac{\text{number of radians}}{2\pi}$$

7. Functions of angles.

An angle is said to be in standard position if, and only if, its vertex is at the origin and its initial ray extends along the positive x-axis. Every angle is equivalent to one and only one angle in standard position. If we place an angle in standard position, e.g., ∠AOB, one of the angles between the terminal ray and a ray of the x-axis must be a positive acute angle, or a right angle, or zero. For any given angle θ in standard position and reference angle is the smallest nonnegative angle between the terminal ray of θ and either ray of the x-axis.



Consider any angle θ, not an integral multiple of 90°, placed in standard position, with its terminal ray cutting the unit circle at P.

$$\sin\theta = \text{ordinate of } P$$

$$\cos\theta = \text{abscissa of } P$$

$$\tan\theta = \frac{\text{ordinate of } P}{\text{abscissa of } P}$$

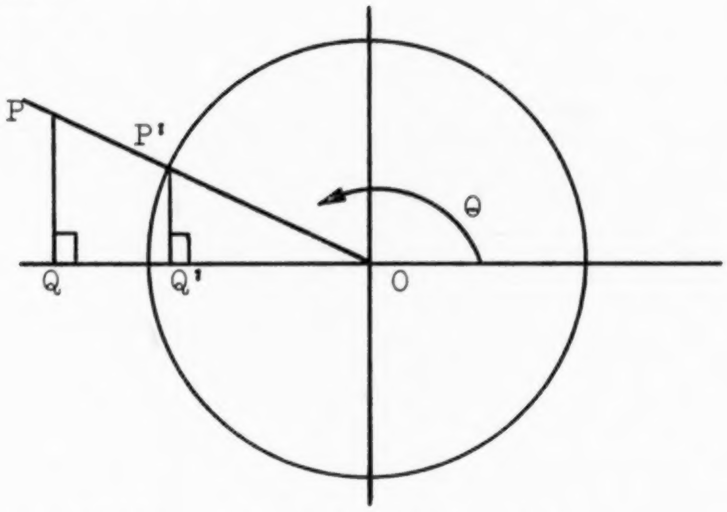
If the angle θ, not an integral multiple of 90°, is placed in standard position, and its terminal ray does not cut the unit circle at P we define the functions as follows: P is any point on the terminal ray.

$$\sin\theta = \frac{\text{ordinate of } P}{\text{polar distance of } P}$$

$$\cos\theta = \frac{\text{abscissa of } P}{\text{polar distance of } P}$$

$$\tan\theta = \frac{\text{ordinate of } P}{\text{abscissa of } P}$$

That these two definitions are consistent may be seen from the diagram below.



Since $\triangle OPQ \sim \triangle OP'Q'$, the following ratios are equal.

$$\frac{PQ}{OP} = \frac{P'Q'}{OP'} = \sin\theta$$

$$\frac{OQ}{OP} = \frac{OQ'}{OP'} = \cos\theta$$

$$\frac{PQ}{OQ} = \frac{P'Q'}{O'Q'} = \tan\theta$$

8. Results of Definition.

The results of the last section lead to the following conclusions for functions of angles in the four quadrants

sin θ - positive cos θ - negative tan θ - negative	sin θ - positive cos θ - positive tan θ - positive
sin θ - negative cos θ - negative tan θ - positive	sin θ - negative cos θ - positive tan θ - negative

Exercises:

1. In which quadrants may the angle θ terminate if

   (a) tan θ is negative.

   (b) cos θ is positive.

2. In which quadrant does the angle θ terminate if
   (a) $\sin \theta$ is negative and $\cos \theta$ is positive.
   (b) $\tan \theta$ is positive and $\cos \theta$ is negative.

3. Find the values of $\cos \theta$, and $\tan \theta$ if $\sin \theta = \frac{3}{5}$ and θ is in quadrant III.

9. Numerical values of functions in any quadrant.

Review the properties of the 30-60-90 degree triangle and the isosceles right triangle. Apply to such problems as the following:

(a) Find the numerical value of $\cos 120^\circ$.

(b) Find the numerical value of $\tan(-150^\circ)$.

(c) Solve the equation $\tan \theta = 1$, θ in quadrant III.

10. Graphs of functions.

The graph of $y = \sin \theta$ may be plotted from a table of values.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{9\pi}{4}$	$\frac{11x}{6}$	2π
$\sin \theta$	0	.5	.7	.9	1	.9	.7	.5	0	-.5	-.7	-1	-.9	-.7	-.5	0

The periodicity of the curve may be observed when the curve is extended for values of θ greater than 2π and for negative values of θ.

$y = \cos x$ - Same treatment
$y = \tan x$ - Emphasize points of discontinuity

11. Trigonometry of the right triangle.

Definitions of the functions for $0^\circ < \theta < 90^\circ$ have been established on the system of coordinates. Next, we redefine the functions in terms of the sides of a right triangle (coordinate free) emphasizing the consistency of the two definitions.

Use these new definitions in the analysis and solution of a variety of problems which can be found in any standard text.

12. NOTE.

It has been suggested that the above unit may not be fully covered because of time limitations. Moreover, there is some question as to whether so extensive a unit is necessary as part of the background that all college-bound students should have at this level. An alternative treatment, much shorter in scope and in teaching time would start with the general definitions of the functions restricted to quadrant I. The definitions can then be applied to the functions of acute angles of a coordinate free right triangle.

GRADE 9 - CHAPTER 10

TANGENCY

Purpose:

To develop and formalize the traditional line tangent to a circle and circle tangent to a circle concepts.

To extend the intuitive concept of tangency for "straights" and curves in both two and three space.

To develop a broader intuitive concept of lines tangent to a curve through the use of tangent envelopes.

Background: yes

Rationale:

The rationale for much of the content in this chapter is primarily explorative and thought provoking. It should present a wide new range of extremely fertile ideas and provide some concrete experience background for many of the ideas of analytic geometry, differential calculus, and advanced geometries.

The rationale for placement of this chapter late in the 7-9 sequence is twofold: there is a relative maximum of mathematical methods and background to be used and there will soon (10-12) be further work of a more definitive nature with these ideas.

Section 1. Circles and line tangents.

1.1 The tangent envelope of a circle.

It is suggested that this idea be started with a paper folding exercise in class:

Exercise: Take a sheet of paper with a circle and its center drawn. Fold the paper so that a point of the circle is on the center. Repeat this process with many other points on the circle.

Result: This will give some members of the tangent envelope of the circle that is concentric with the given circle and has a radius that is one-half the given radius.

Center the written discussion on the idea that while no new circle is actually formed by this process (a polygon does evolve), there is a feeling for a circle. Perhaps have the students draw that circle (with compasses). Discuss how the folds are related to this new circle.

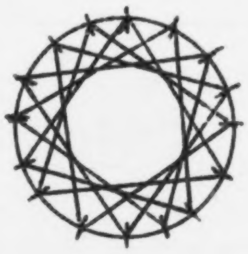
Exercises:

1. Give a circle with points indicated on the circle every 10°. Label the points 0, 1, 2, ..., 35. Have the students join the points by a mapping which is of the form:

   $n \to n + a \pmod{36}$ where n is an integer, $0 \leq n \leq 35$, a is a parameter with integral values.

   Note: Students catch on to the modulus system quickly through reference to the clock.

   Example: $n \to n + 5$



Could the students draw the determined circle? Perhaps several different examples might be given of this form. Aside: n $\rightarrow$ n + 6 (mod 36) gives the same result as n $\rightarrow$ n - 30 (mod 36). Would any student note this -- or care?)

2. Give a circle and a point exterior to the circle. Have the student draw many members of the family of lines through the given point. Ask for some thought (and perhaps written discussion) about how these lines are related to the given circle. Probably they can note: one line through the center (hopefully not three points of intersection); many lines through two points of the circle but not the center; two lines through exactly one point of the circle; many lines that do not intersect the circle.

1.2 Lines Tangent to a circle.

Define secant and line tangent to a circle.

Define point of tangency.

Property: A line tangent to a circle is perpendicular to the radius drawn to the point of tangency.

Exercises:

1. How many lines tangent to a given circle can be drawn through a given external point? Why?
2. ... through a given internal point? Why?
3. ... through a given point on the circle? Why?
4. If a line is perpendicular to a diameter at one of its endpoints, is the line tangent to the circle? Prove your answer.

Note: they should be able to do this if the chapter on deductive reasoning was successful!

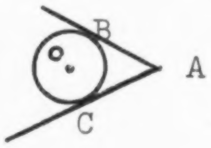
5. A point is 8 units from a circle of radius 5 units. A line is drawn through the point tangent to the circle. What is the distance from the given point to the point of tangency?

6. A line tangent to a circle is drawn from a given point. The distance between the given point and the point of tangency is 8 and the radius of the circle is 5. What is the distance of the given point from the center of the circle? from the circle?

7. Draw a circle. Draw three lines that are tangent to the circle such that no two of the lines are parallel. Describe the resulting figure.

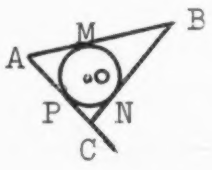
Note: this problem might be repeated for 4, 5, ... lines.

8. Given: $\overleftrightarrow{AB}$ tangent to circle O at B
$\overleftrightarrow{AC}$ tangent to circle O at C



Prove: $\overline{AB} \cong \overline{AC}$
$\overrightarrow{AO}$ bisects ∠BAC

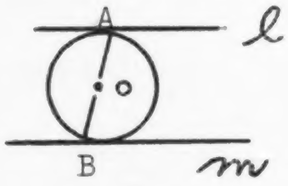
9. Given: $\overleftrightarrow{AB}$ tangent to circle O at M
$\overleftrightarrow{BC}$ tangent to circle O at N
$\overleftrightarrow{AC}$ tangent to circle O at P
AB = 8, BC = 7, AC = 11



Find BM, BN, NC, CP, PA, AM

Note: This might be extended to a quadrilateral etc.

10. Given: circle O with diameter AB
$\ell \perp AB$ at A, $m \perp AB$ at B



Prove: $\ell \parallel m$

1.3 Constructing tangents.

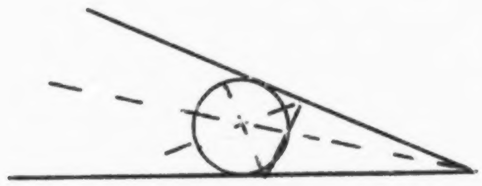
Develop:

(1) construct a line tangent to a given circle at a point on the circle.

(2) ... through a point external to the circle (note that exactly two are determined)

(3) Construct a circle inscribed in a given triangle.

(4) Construct the three excircles of a given triangle (the excenter is the point of concurrency of the angle bisectors of one interior angle and two exterior angles) See Intro. to Geometry by Coxeter, pp. 11-12.



(5) Construct some of the members of the family of circles tangent to a given line at a given point on the line.

(6) Construct some of the members of the family of circles tangent to a given line and through a given point not on the line.

(7) Construct some of the members of the family of circles tangent to each of the sides of a given angle.

Challenge Problem: The Nine Point Circle
See Coxeter pp. 18-19. 71

Note: this is highly constructable but might it spoil this gold mine for later work?

1.4 Angles formed by tangents.

Should the measures of certain angles (tangent/chord, tangent/secant, etc.) be developed at this point?

Section 2. Tangent lines and planes in two and three space.

Short discussion of what is meant by "a line tangent to a curve", "a line tangent to a surface", and "a plane tangent to a surface". Keep it *light*.

Exercises:

A set of highly intuitive, discussion provoking problems that require visualization and some verbalization. Might try such things as:

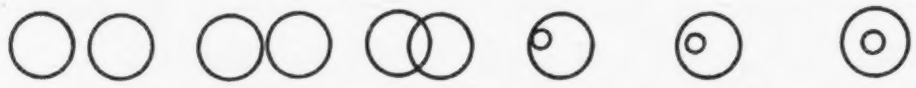
1. A line tangent to: a parabola; a sine curve; a tangent curve; a given polynomial curve.
2. A line tangent to: a sphere; a conical surface; a cylindric surface; a torus.
3. A plane tangent to: a sphere; a conical surface; a cylindric surface; a torus.

Make these very open ended and provide help for the teacher in the teachers manual as to method of handling and expectations.

Section 3. Circle and line tangencies extended:

3.1 The relationships of two circles.

Discuss all cases:



Exercises: See SMSG Geometry Chapter 13.

3.2 Three on more tangent circles.

Develop relative to two different conditions: tangent by pairs and all tangent at one point.

Exercises: See SMSG Geometry Chapter 13.
See also Coxeter Intro. to Geometry
Be sure to include families of tangent circles and conditions that lead to these.

Section 4. Tangent plane curves and tangent curved surfaces:

If this is handled WSTWSTWST, it might be quite interesting; e.g., two tangent sine curves. However, it might be too difficult to establish what is meant by "tangency" under such conditions. It should be worth a try on a highly intuitive level -- just to assure that the students have at least tried to think about such things.

Might consider such things as:

(a) two tangent parabolas at some point other than the vertices,
(b) a family of parabolas tangent at the vertices,
(c) the family of curves of the form $y = x^{2n}$, $n \in I$
(d) $y = \frac{1}{x}$ and $x^2 + y^2 = 1$,
(e) two tangent ellipses ... or families,
(f) two tangent spheres ... or families,
(g) a cylinder tangent to a sphere (aha ... Mercator),
(h) a cone tangent to a sphere (aha ... Lambert).

Section 5. Tangent Envelopes:

Teaching note: do not have long discussions of these -- EXPLORE. Each sub-section (5.1, 5.2, 5.3) might easily be done in one day. If the students get intrigued, let them change the given conditions and see what happens on their own.

5.1 The conics.

(a) Parabola: given a line and a point not on the line.
Fold the given point onto one point of the line.
Repeat for many other points of the line.

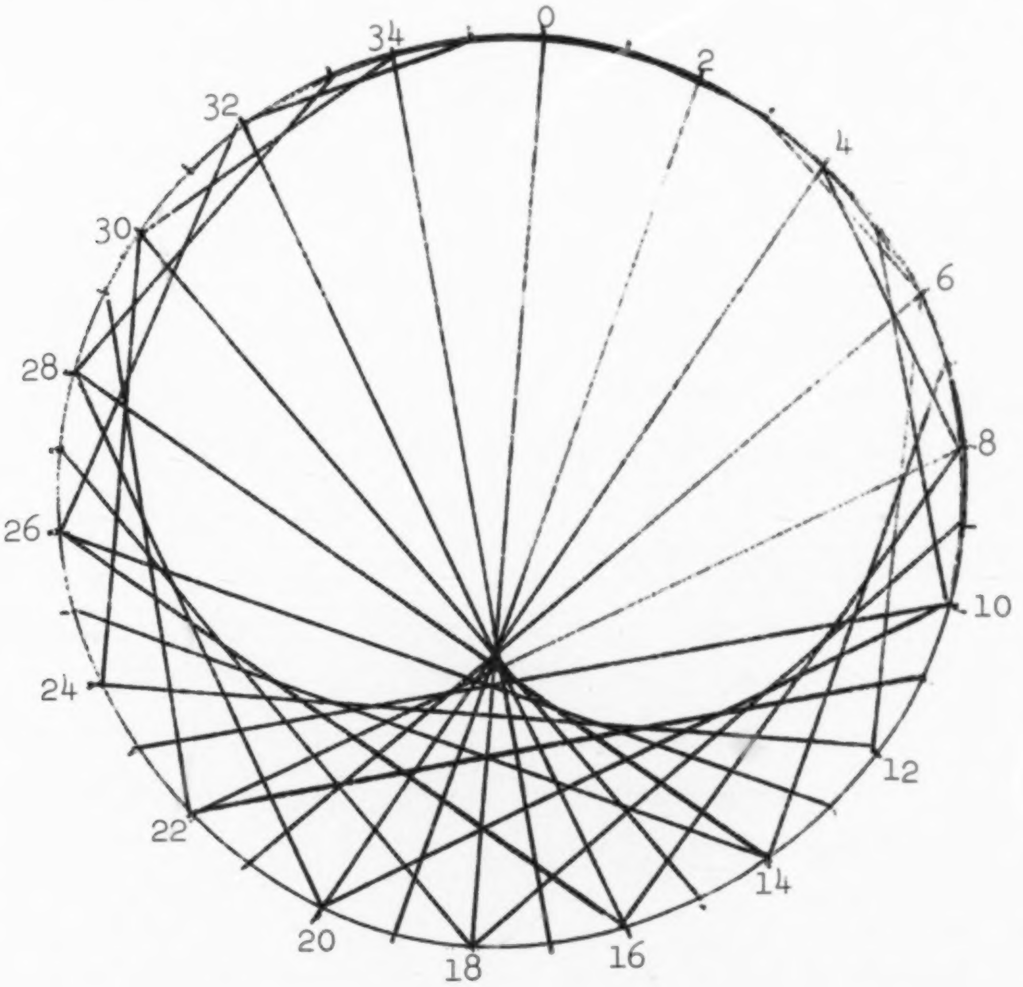
(b) Ellipse: Given a circle and a point interior to the circle. Fold the given point onto one point of the circle. Repeat for many other points of the circle.

(c) Hyperbola: Given a circle and a point exterior to the circle. Fold the given point onto one point of the circle. Repeat for many other points of the circle.

5.2 Joining points on a circle.

Several interesting "twists" can be put on Problem 1 of Section 1.1.

Example: n → 2n (mod 36)



Extension: $n \rightarrow an + b \pmod{36}$

Some students will really carry this a long way if plenty of labeled circles are provided to help minimize the "busy work".

5.3 Working in a co-ordinate system.

Problem: Consider the set of lines through $(a, \frac{a^3}{3})$ with slope a^2.

Directions: Draw some of these lines using at least these values for $a : -3, -\frac{3}{2}, -2, \ldots, \frac{3}{2}, 3$.

Teachers note: This gets the tangent envelope for the curve whose equation is $y = x^3$ so it included one example of a line tangent to a curve at a point of inflection. Don't let this be missed.

5.4 Pursuit Curves.

Note: Should these be included since the others have actually been members of the tangent envelope to one curve under consideration?

(a) Dog who "re-aims" every 2 sec. is chasing a rabbit who runs in a straight path.

(b) Same rabbit but smarter dog -- he "re-aims" every second.

(c) Rabbit who runs in a circle where the dog (who re-aims every 2 sec.) is at the center.

Section 6. Line tangents to any curve.

Work from the preceding tangent envelopes toward some understanding of the tangent to a curve at a given point in terms of the limiting position of the secants through the point. Give lots of curves and stress looking for the point. Give lots of curves and stress looking for the entire tangent envelope -- then pick out some "interesting" points on the curve to look at specific members. The student should be able to develop some feeling in terms of tangents relative to curvature, cusp as opposed to turning point and points of inflection. Take what the student is sensing, and discuss in terms of the secants at each point. Be sure the student develops some ability to verbalize relative to these things. The brave of the bold or the fool-hardy might try a discussion of the tangents to $y = |x|$.

Note: this section is very important. Section 5 "brings on the cannon" but Section 6 fires it!

Section 7. Line of support:

See some material on linear programming -- this is a somewhat different sense of tangency but might be discussed at this point.

GRADE 9 - CHAPTER 11

MEASURE

Very brief outline:

1. Use, in parts, material found in Geometry, SMSG, Volume 2, Chapters 12, 15 and 16. It can be exprected that not much of this material will find its way into the 10th grade semester course of geometry.
2. Measure was, in the following order, treated earlier:

   Grade 7 : Chapter 5
   Grade 8 : Chapter 11
   Grade 9 : Chapter 4 (measure theory).

Concentrate in this round of measure on developing further the understanding of the concept of similarity by providing examples of linear, quadratic and cubic measures.

Example: What are the changes in the perimeter, area, volume of an object if a dimension is doubled, halved, etc.?

Try to aim at theorems, informally only, to show their power: we don't have to compute the surfaces or volumes of two similar solids in order to compare them.

Start possibly with: How many geometric objects can you name? What formulas do you know?

Make a list (with some ordering principle):

Name	Diagram	Dimensions	Perimeter	Area	Surface	Volume
	(well labelled)			which ever applies.		

The purpose of this would be three-fold:

(1) Motivation: this may become more and more of a problem when review does not contain some ingredients of preview, and topics have appeared repeatedly.

(2) Review and summary.

(3) A ready list of reference to use in the subsequent discussion -- the formula should be more than a recipe to get answers; this can become an important link in bringing algebra and geometry together.

Include the following questions:

(A) 1. Given a square region of side s; double the side. What is the effect on p, on A? Make a diagram.

2. Given a 30-60-90 triangle; multiply sides by 3. What happens to the altitude and median upon the hypotenuse? To the perimeter? To the area? Repeat for either values: Multiply the sides by 5, by $\frac{1}{2}$, by $\frac{1}{4}$, by n. Make a diagram.

3. The radius of a circle is halved. What is the effect on c, on A? Can you tell from a diagram in this case as much as you could in Exercise 1 and 2?

Generalize: Do we have to carry out the computation or can we draw conclusions? What will hold generally?

(B) Next change the questions to:

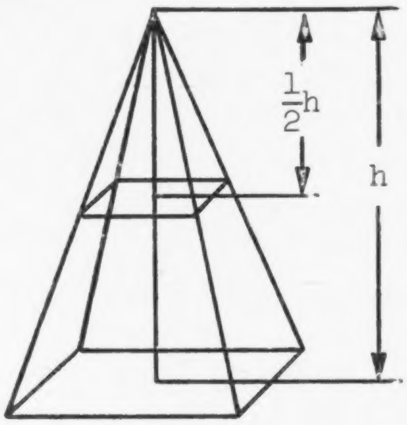
1. How must the edge of a cube be changed if it is to be eight times as large? be twice as large in volume?

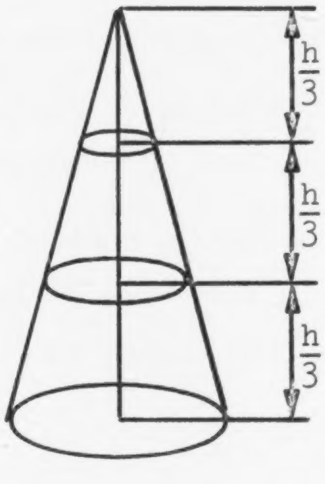
2. A circular region is to be divided into 3 equal parts by concentric circles. How can this be done?

(C) Before turning to volumes, contrast the relation of linear measures and areas of similar figures to questions of the type, (emphasis on analysis of the algebraic formulas):

1. Given a triangle with base c and height h. What is the effect upon the area if the base is multiplied by 3 and the height halved? If the base and the height are multiplied by 5?
2. The radius of a circle is multiplied by 5. What happens to the area? to the circumference?
3. What happens to the volume of a cylinder (r,h) if the height is doubled? If the radius is doubled? If the height is halved and the radius doubled? If the radius is halved and the height doubled?
4. Which will make the volume of a cone larger, doubling the radius or doubling the height?

Now discuss similar solids, produced by passing a plane parallel to plane of base.





By comparing:

(a) edges, altitudes, radii, etc.

(b) areas of cross sections, lateral faces, etc.

(c) volumes of similar solids,

develop now systematically the relation

$$\frac{p_1}{p_2} = \frac{a}{b}\,, \quad \frac{A_1}{A_2} = \left(\frac{a}{b}\right)^2, \quad \frac{V_1}{V_2} = \left(\frac{a}{b}\right)^3.$$

NOTE I: It may seem wiser to complete the work on measure with the "WST" development of surface and volume of the sphere now, rather than present it in Grade 8, Chapter 11. This would leave for Grade 8, The Earth as a representation of a sphere and allow more time to discuss more fully:

great circles
small circles
location of a point on the earth
measurement along a meridian
time difference,
etc.

NOTE II: In the original outline appears: Pythagorean relationship and trigonometry.

The g.c. feel that this should be omitted at this point.

However, the following suggestion is for inclusion when this topic is handled in its proper setting:

Given a right triangle and

(A) any two parts (sides), the third side can be computed by the Pythagoream Theorem (no angles can be determined).

(B) any two parts (one side and one acute angle, or two sides), the triangle can be determined completely by trigonemetric functions; sides and angles can be computed.

SUMMARY OUTLINE, GRADES 7-9

Grade 7

Chapter 1 *Non-metric Geometry - The Structure of Space*

Point, line, plane. Incidence. Separation. Convexity. Orientation on a line, in a plane.

Chapter 2 *Graphs, Functions, Variables*

Coordinates. Function. Graphs of functions.

Chapter 3 *The Set of Rationals - Solution of Mathematical Sentences*

Definition of rational number. Addition and subtraction of rationals. Decimal names for rationals. Ordering the rationals. Per cent. Solving equations and inequalities.

Chapter 4 *Congruence - Replication of Figures*

Congruence of segments, of angles. Addition property for segments. Subtraction property for segments. Addition and subtraction property for angles. Vertical angles. The concept of congruence. Congruence of a figure with itself. Congruence of triangles. SSS congruence property. SAS. ASA. Motions by means of a coordinate system.

Chapter 5 *Measure*

Linear units. Angular and arc measure. The Pythagorean property and applications. Equivalence of polygonal regions. "Greater than" for segments, angles, planar regions, spatial regions.

Chapter 6 *Ratio and Similarity*

Magnification and contraction. Concept of similarity. Ratio and proportion. Defining similarity. Sufficiency properties for triangles. Similarity mappings.

Chapter 7 *Probability*

Fair and unfair games. Finding probabilities. Counting outcomes - Tree diagrams. Pascal's triangle. Estimating probability by observation. Organization of data leading to average and expectation. $P(A \cup B)$. $P(A \cap B)$.

Chapter 8 *Graphs of Linear Functions; Multiplication of Rationals*

Review of negative rationals. Multiplication of positive by negative. Graphs of multiplication by positive rationals.

Multiplication of positive by negative and Distributive Law. Multiplication by negative rationals. Addition and subtraction revisited. Opposite function; absolute value function. Applications. Graphing $x \rightarrow ax + b$.

Chapter 9 Solutions of Systems of Equations and Inequalities

Solving systems of equations. Systems which do not have unique solutions. Graphs of inequalities. Systems of inequalities.

Chapter 10 Decimals, Square Roots, Real Number Line

Motivation. Numbers which are not rational. Names of rational numbers. Irrational numbers. Real number line. Properties of real number system.

Chapter 11 Parallelism

Parallel one-dimensional objects, two-dimensional objects. Transversals. Transversals to three or more lines and planes.

Grade 8 (Sequence A)

Chapter 1 Perpendicularity

Perpendicularity of one-dimensional objects, of two-dimensional objects.

Chapter 2 Coordinate Systems - Distance

One-dimensional coordinate system. Two-dimensional coordinate system. Three-dimensional coordinate system. Polar coordinate system.

Chapter 3 Displacements (Vectors)

Quantities. Vector quantities. Vectors. Physical multiplication of vectors by a number. Translation. Decomposition. Applications. Extension to fectors in three-space.

Chapter 4 Problem Analysis

Translation of phrases. Translation of sentences. Problem analysis and strategies.

Chapter 5 Number Theory

Even and odd integers. Informal discussion of statements and proof. Factors, divisibility, tests for divisibility and the division algorithm. Prime numbers, the sieve of Eratosthenes, prime factorization. The Euclidean algorithm and the GCD.

Chapter 6 The Real Numbers Revisited - Radicals

Motivation. Review of facts about the real number system. Roots of numbers. Computation with radicals. Review of properties of the real numbers and the real number line.

Chapter 7 Truth Sets of Mathematical Sentences

Review addition and multiplication properties of equality and inequality. Apply to inequalities and to problems. "Permissible Operations." If $a = b$, then $a^2 = b^2$ and converse. Fractional equations and restrictions on denominator. Squaring both sides of equation and equivalent equations.

Chapter 8 Quadratic Polynomials as Functions

Graphing a quadratic function. Factoring polynomials. Solving quadratic equations. Going from $ax^2 + bx + c$ to $a(x-h)^2+k$.

Chapter 9 Probability

Dependent and independent events. Conditional probability. Bayes' Theorem. Expectation. Variation, standard deviation. Normal distribution. Physical observations.

Chapter 10 Parallels and Perpendiculars

Regions. Combining parallel and perpendicular relations. Distance between parallel lines and parallel planes. The quadrilateral properties. Symmetries. Angle sum proofs.

Chapter 11 Properties and Mensuration of Geometric Figures

Motivation of numerical measure for areas. Arbitrary unit versus standard unit. Assigning measure to segments and regions. Properties of regular polygons. Models of solids. The sphere.

Chapter 12 Spatial Perception and Locus

Relationships between two or more point sets. Using a set of points to evolve another set of points. Sets of points meeting given conditions.

Chapter 13 Systems of Equations in Two Variables

Solution sets of systems of equations and inequalities. Equivalent equations; equivalent systems. Systems of linear equations. Graphical solutions of systems of inequalities. Applications.

Grade 9

Chapter 1 Exponents, Logarithms, Slide Rule

Laws of exponents. An exponential function, $f:n \rightarrow 2^n$. Computation using powers of 10. Introduction of log notation. Slide rule construction and use. Exponential and logarithmic functions.

Chapter 2 Transformations

Rigid motions and reflections. Projection. Composition of transformations. Congruence as an isometric correspondence. Similarity as a ratio preserving correspondence. Further work on symmetry.

Chapter 3 and 4 Systems of Sentences

Chapter 5 Measure Theory

Distance. Measure. Angle measure. Other measures.

Chapter 6 Statistics

Organization of data - Histrograms. Mean. Variance. Confidence intervals for mean. Hypothesis testing. Binomial theorem. Normal distribution. Central limit theorem.

Chapter 7 Deductive Reasoning

Illustrations of logical relationships between statements. Suggestions for geometric examples. Illustrative problems.

Chapter 8 Vectors

Chapter 9 Circular Functions

Periodic motion. Sine, cosine, tangent functions. Domain and range. Circular functions and angles. Radian measure. Functions of angles. Numerical values of functions. Trigonometry of the right triangle. Graphs of functions.

Chapter 10 Tangency

Circles and line tangents. Tangent lines and planes in two- and three-space. Tangent plane curves and tangent curved surfaces. Tangent envelopes. Line tangents to any curve. Line of support.

Chapter 11 Measure

Chapter 12 Complex Numbers

COMPARISON OF THE NEW OUTLINE AND THE ORIGINAL

Mathematics for Junior High School, Volume 1

Chapter and Topics		Location in New Outline
Chapter 1	What is Mathematics?	No comparable chapter.
Chapter 2	Numeration	No chapter.
2.2	The Decimal System	Gr. 7, Ch. 10
2.3	Expanded Numerals	Gr. 7, Ch. 3
	(Other number bases)	No mention.
Chapter 3	Whole Numbers	No explicit treatment.
Chapter 4	Non-Metric Geometry	Gr. 7, Ch. 1
Chapter 5	Factoring and Primes	Gr. 8, Ch. 5
Chapter 6	The Rational Number System	Gr. 7, Ch. 3, 8
Chapter 7	Measurement	Gr. 7, Ch. 5 Gr. 8, Ch. 11 Gr. 9, Ch. 5 and 11
Chapter 8	Area, Volume, Weight, Time	Gr. 8, Ch. 11
Chapter 9	Ratio, Percents, and Decimals	Gr. 7, Ch. 3, 6, 10
Chapter 10	Parallel, Parallelograms, Triangles, Right Prisms	Gr. 7, Ch. 11 Gr. 8, Ch. 10
Chapter 11	Circles	Gr. 7, Ch. 5 Gr. 8, Ch. 11 Gr. 9, Ch. 10,11
Chapter 12	Mathematical Systems	No chapter.
Chapter 13	Statistics and Graphs	No comparable chapter. Gr. 9, Ch. 6
Chapter 14	Mathematics at Work in Science	No comparable chapter.

Mathematics for Junior High School, Volume 2

Chapter and Topics		Location in New Outline
Chapter 1	Rational Numbers and Coordinates	Gr. 7, Ch. 3, 8 Gr. 7, Ch. 2 Gr. 8, Ch. 2
Chapter 2	Equations	Gr. 7, Ch. 3 Gr. 8, Ch. 4
Chapter 3	Scientific Notation, Decimals, Metric System	Gr. 9, Ch. 1 Gr. 8, Ch. 11
Chapter 4	Constructions, Congruent Triangles, Pythagorean Property	Gr. 7, Ch. 4, 5, 6
Chapter 5	Relative Error	No treatment.
Chapter 6	Real Numbers	Gr. 7, Ch. 3, 10 Gr. 8, Ch. 6
Chapter 7	Permutations and Selections	No treatment.
Chapter 8	Probability	Gr. 7, Ch. 7
Chapter 9	Similar Triangles and Variation	Gr. 7, Ch. 6, 8
Chapter 10	Non-Metric Geometry	No comparable chapter.
Chapter 11	Volumes and Surface Areas	Gr. 8, Ch. 11 Gr. 9, Ch. 11
Chapter 12	The Sphere	Gr. 8, Ch. 11 Gr. 9, Ch. 11
Chapter 13	What Nobody Knows About Mathematics	No comparable chapter.

First Course in Algebra

Chapter and Topics		Location in New Outline
Chapter 1	Sets and the Number Line	Gr. 7, Ch. 3
Chapter 2	Numeral and Variables	No chapter.
Chapter 3	Sentences and Properties of Operations	Gr. 8, Ch. 7, 13 Gr. 7, Ch. 9
Chapter 4	Open Sentences and English Phrases	Gr. 8, Ch. 4
Chapter 5	The Real Numbers	Gr. 7, Ch. 10 Gr. 8, Ch. 6
Chapter 6	Properties of Addition	No specific chapter.
Chapter 7	Properties of Multiplication	Gr. 7, Ch. 8 No comparable chapter.
Chapter 8	Properties of Order	Gr. 7, Ch. 10 Gr. 8, Ch. 6
Chapter 9	Subtraction and Division for Real Numbers	Gr. 7, Ch. 10
Chapter 10	Factors and Exponents	Gr. 8, Ch. 5 Gr. 9, Ch. 1
Chapter 11	Radicals	Gr. 8, Ch. 6
Chapter 12	Polynomial and Rational Expressions	No polynomials. Quad. Gr. 8, Ch. 8
Chapter 13	Truth Sets of Open Sentences	Gr. 7, Ch. 3 Gr. 8, Ch. 7
Chapter 14	Graphs of Open Sentences in Two Variables	Gr. 7, Ch. 2, 8
Chapter 15	Systems of Equations and Inequalities	Gr. 7, Ch. 9 Gr. 8, Ch. 13
Chapter 16	Quadratic Polynomials	Gr. 8, Ch. 8
Chapter 17	Functions	Gr. 7, Ch. 2 Gr. 8, Ch. 8

ON APPLICATIONS

Clyde L. Corcoran

This "second round" planning group was given, as one of its tasks, the problem of imparting to the student some understanding of the role of applications of mathematics to the real world. This was supposed to be done for the students in a meaningful way. I submit that we have not really faced this problem squarely as yet, and there are some good reasons why this is so.

1. It is easy to say, "Let's get the sequence of mathematical topics organized, and then the writers can illustrate the use of the ideas where appropriate." This, I claim, will result in artificial situations of little value.

2. Real "practical" applications of mathematics do not tend to be permanent entities. For example, while applied problems in percent might be valuable to the present suburban resident, they really could be termed vital to a pioneer crossing the plains a hundred years ago. The mathematics of the "honeycomb" was interesting to a very few twenty or thirty years ago, but now this interests a whole spectrum of individuals concerned with lightweight metal construction and fusion processes.

3. Most meaningful applications require extensive backgrounds in subject areas other than mathematics. Most students do not have these extensive backgrounds and hence understand neither the mathematics involved nor the application. Usually the situation which is described in a few sentences is so artificial or trivial that the student is not interested and is unable to see any use of it in the real world.

I submit that we really do not wish to teach "applications" as such but that we really should try to expose the student to the techniques of applying mathematics. I'm speaking of the techniques and reasoning processes which allow a professional mathematician to analyze and solve problems in many diverse fields of study without really being expert in those fields.

I also submit that we should make clear to the student that there are at least two categories of important applications of mathematics.

One is the application of mathematics to itself (i.e., algebra to geometry) to derive additional mathematics, and the other is the application of mathematics to other fields of study. I feel that the "internal" application of mathematics was slightly exposed in the first round SMSG but that it is important enough to be explicitly pointed out as it occurs in this second round development.

If we accept the hypothesis that we should concentrate on teaching the techniques of applying mathematics to "practical" situations rather than trying to teach specific techniques for handling certain practical problems, then we should devise a program which will at least illuminate these procedures.

A possibility for one such program could be as follows: It might be possible to create a series of units for each grade level which would allow the teacher to select and present an "application problem" in depth. The major purpose of this approach would be to expose the techniques for applying mathematics rather than try to say that we are teaching mathematics applied to biology, home economics, mechanics, economics, chemistry, and so on.

It is hoped that problems within the ability of the students could be devised so that the student would experience many of the same procedures that mathematicians would use in tackling a problem. Also it is hoped that many different parts of mathematics would enter naturally into the analysis and solution of the problem (arithmetic, algebra, geometry, probability, linear algebra, etc.). I would hope that the student would experience the necessity of having to clarify the problem (decide what the basic questions are), plan methods of attack, search out and collect data, and organize that data to reveal information about the problem, as well as develop models of the problem as needed.

Many of the situations in the Mathematics Through Science series or in Mathematics and Living Things might be redesigned to accomplish the above objectives. Some linear programing problems like the standard "diet" problem also might have some possibilities for this kind of exposition. The seventh grade problems could be highly structured, but eighth and ninth grade problems could leave more to the student's originality and creativity with some open-ended questions included to forestall the impression that all problems can be answered.

To summarize, I think that we are on firmer ground if we attempt to teach basic techniques for applying mathematics rather than try to teach specific applications of mathematics. Perhaps we would even satisfy some of the critics who say that these publications are too concerned with formal mathematics rather than mathematics of the "real" world.

SOME COMMENTS ON THE ROLE OF "FLOW CHARTING" IN JUNIOR HIGH SCHOOL MATHEMATICS

S. Sharron

There is no need to list the ways in which the computer has become an essential part of the everyday technological world. Suffice it to say that any student who attends high school today should have the opportunity of exposure to a curriculum that includes some recognition of the role of computers in our technological society.

Having made this premise, one is confronted with the problem of the nature, degree, and location of this topic in the high school curriculum. To understand the operational aspects of a computer is a varied and complex undertaking that is not entirely in the province of a mathematics course. At least an appreciation for the mathematical activities associated with computer use should be part of the high school mathematics program. A student should be able to think in terms of computer problem solving techniques (even if only at basic or introductory levels) if he is to prepare for almost any vocation in a world where there are seemingly, unlimited horizons of activity for the ever increasing use of the computer.

A method of communication between an individual and the computer has been accomplished by the development of procedural languages which have as their purpose the task of relating to the computer some "unambiguous plan telling how to carry out a process in a finite number of steps." The algorithm (as the plan is called) requires a highly sequential step by step method of computing a problem and an early link in this line of communication is the "flow chart" (a first translation from English to diagram). Because of the careful, well disciplined procedure required in preparing a flow chart, a student would of necessity need to have the mathematical algorithm well in mind if he is to complete a successful flow chart. A successful flow chart is one which could be converted to the appropriate machine language for a particular computer and which will enable that computer to complete the problem successfully.

Almost immediately a high school mathematics teacher sees here a real world related method of reinforcing computational skills. Not only does the student become motivated by the idea of computer work, but he must develop a working knowledge of the mathematics involved. Some points seem worth mentioning at this time:

(1) One should bear in mind that the flow chart though less formal than the machine language, requires the student to put down a series of low order steps (with well disciplined care and forethought) for a simpleminded machine. This is the prime purpose of flow charting. It is not intended to be an end in itself, and unless it ultimately ends up in a computer after having been translated to the appropriate machine language with the results for the student to see, the motivational factor for flow charting becomes shaky.

(2) The knowledge of flow chart language notation required for effective computer-oriented work requires preparation and learning time. The SMSG test, Algorithm Computation and Mathematics, concerns most of its size with this objective. A brief presentation directed at a lower level for 7th to 9th graders could be made, but practice and reinforcement is still required if any usable degree of technique is to be achieved.

(3) A flow chart is an early step in computer programming, and the completed program is useful to the computer in that it can be stored and reused whenever needed. Once a successful flow chart has been constructed, it becomes available for repeated use as written communication structured in detail form to serve an assortment of machine languages. For the student who constructs a flow chart, the experience of exploiting a mathematical algorithm may be rewarding

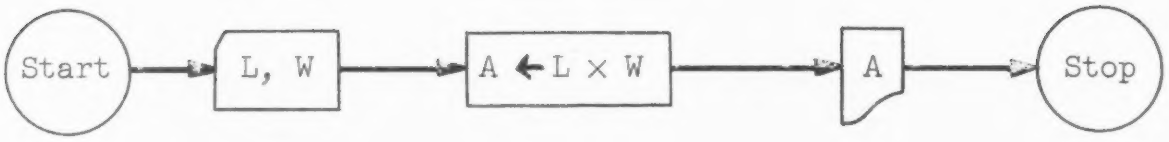
and often he gains insight, but the flow chart itself is not usually a necessary part of the student's working equipment in order for him to function in mathematics. In other words, the flow chart referred to is often employed as a part of the chain of events leading to effective use of a computer.

(4) Many (not all) assumed to be simple mathematical algorithms when flow charted offer a method of primitive manipulation not suitable (though perhaps interesting) for the skills we hope to have the students reinforce, e.g., see the New Orleans report, page 30, Long Addition Algorithm (1) and (2).

It may appear that the points raised are sufficient reason to drop the whole thing. On the contrary, these points are intended to indicate difficulties to be overcome and pitfalls to avoid. The original premise still stands, but it is not intended that the junior high school mathematics program should become a course in computer programming if for no other reason than that flow charting is essentially an application or exploitation of mathematics and is not usually mathematics per se.

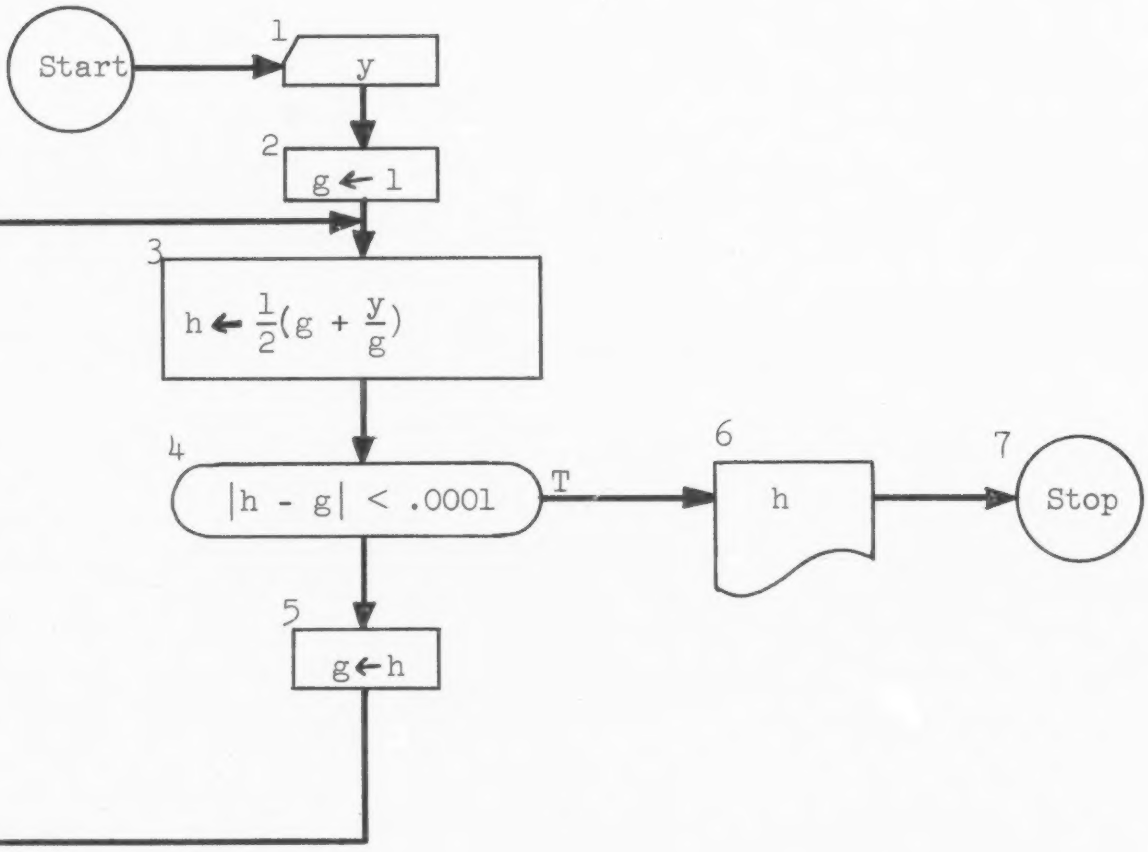
Often in trying to construct a flow chart, one finds a need for more mathematics or a better understanding of the mathematics he already has available. This is desirable in the interest of teaching mathematics and situations spread out through the 7th to 9th grade will occur where some flow chart activity will be effective. For example, the development of the flow chart for the Euclidean algorithm (g.c.d.) as shown in SMSG test, Algorithms Computation and Mathematics, chapter 3, section 2, pages 113-121 offers an excellent opportunity for a student to gain depth in understanding this topic which is part of the proposed 8th grade chapter on number theory.

In 7th grade, the introduction of flow charting could begin with computers (a discussion on what they are and how they are utilized), and the flow chart symbols and variables applied to simple problems such as:



(See SMSG text Algorithms Computation and Mathematics, chapter 2) If during the course of the year the student is introduced to a selected number of flow chart experiences (little more than one per chapter starting with flow charting as one example of modeling), he should be ready to reap more of the benefits in 8th grade offered by flow charting problems like the Euclidean algorithm, making decisions (is $A > B$?, if so, what then, or if not, what then?), and looping (to do iterative processes). Again in 8th grade the activity should be a limited number of selected problems giving preference to those appropriate for understanding the mathematics in the 8th grade program over the techniques required for programming.

It would appear that flow charting the long addition, subtraction, multiplication and division algorithms are too involved in computer orientation to give the best returns for mathematical benefit to the student. Another caution to be exercised in the selection of flow chart activities can be illustrated by the following example. Grade 7, chapter 10, (proposed) introduces a method of approximating $\sqrt{5}$ to a specified number of digits by an iteration method. The flow chart for this process shown in chapter 5, section 1, pages 223-225 of the earlier referenced text, is an excellent opportunity for the student to gain a thorough understanding of the Newton method,

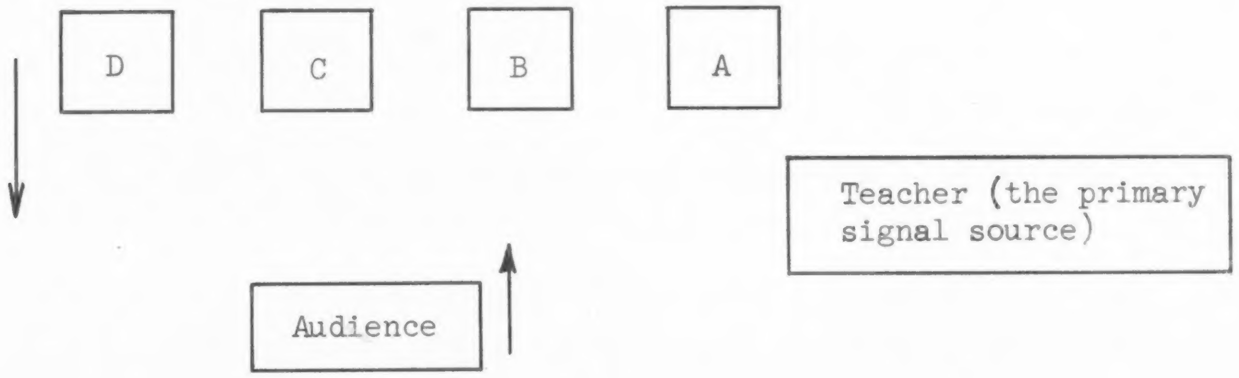


but the mathematics used in getting assignment box 3 is more advanced than his grade level. If a flow chart activity is to be used at this time, consideration would have to be given to the student's frame of reference.

Early in the 7th grade the teacher should introduce the uses to which computers have been put in industry, a brief history of how computers have developed, and a look into the future. This is the preface suggested to the introduction of flow charting. The SMSG text, Algorithms, Computation and Mathematics hereafter referred to simply as the ACM text offers considerable data along these lines in Chapter I, Section 1-1, 1-2; which could be reworded and made more suitable for 7th grade.

An interesting activity based on an idea by Engelbart provides a game atmosphere along with educational results that tends to dissipate the mystery associated with computers.

Four children are arranged in a line in front of the class from left to right.



They are told that their right hand must be clearly up or down in doing what follows. The child, D, on the extreme left is told that his signal comes only from the child, C, on D's immediate left. The signal is C dropping his hand from up to down. At such a signal from C, D changes the position of his right hand. Child C gets his signal only from child B on C's immediate left and the signal is the same; that is, C accepts a signal only from B which takes place when B drops his hand from up to down. Upon receiving such a signal from B, C changes the position of his hand. Similarly B gets his signal when A drops his right hand from up to down. Notice that raising a hand from down to up is no signal. Child A is the only one who gets a signal from the teacher, by means of a hand clap or a finger snap.

Practice may be necessary before the teacher begins to signal. The teacher begins the first signal after having each hand in the down position. Notice that after 16 pulses (teacher signals) all the hands are in the down position again.

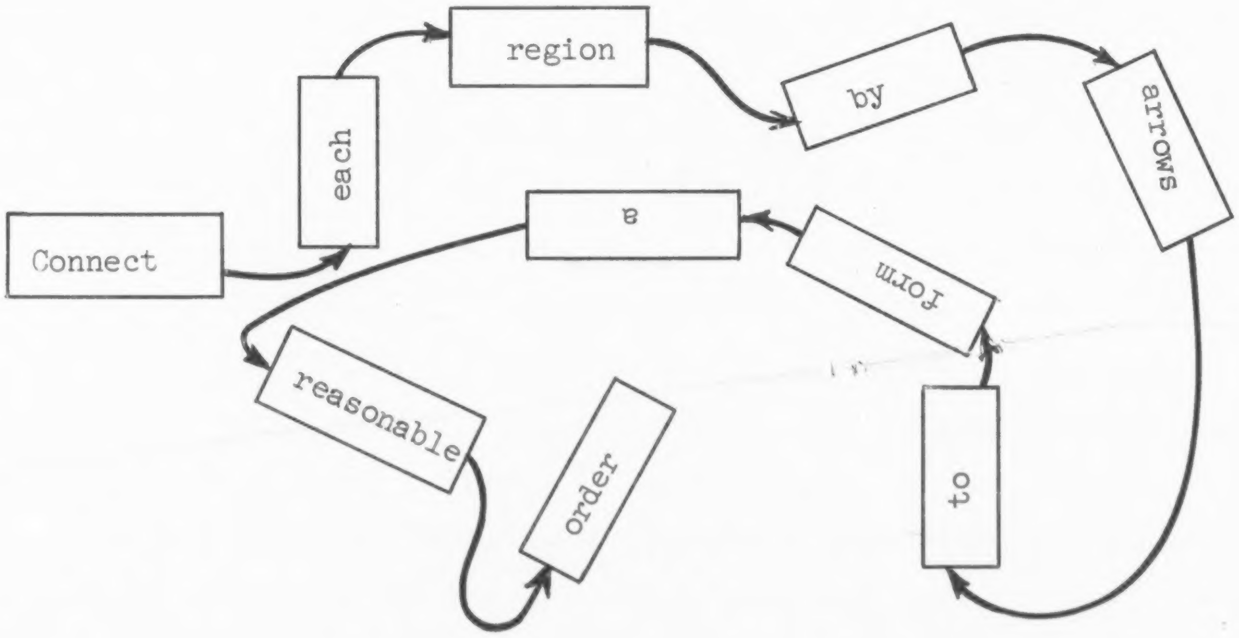
When the four element system seems to be functioning smoothly, the children are given cards labelled 1, 2, 4 and 8 distributed to A, B, C, and D respectively. The pulses begin again and is interrupted occasionally to ask how many signals have been given. Each time, the class will notice that the number of signals given is equal to the sum of the numbers held up. If one is interested in pursuing the binary system, the class may be given the opportunity to figure out how specific numbers (which cards to be held up) should look. Also, the limit of the counter can be increased to 31 by adding another element (child).

One interesting activity that may be performed is to have the teacher signal enough pulses to enter a number, say, 5 on the computer. Having done that, he can add another number, say, 6 to the system. The result will be cards held up in which the numbers will total 11. The last activity can be varied; but taking into consideration the limitations of the machine, the class can see from this demonstration that the teacher, by his selection of pulses, actually programs the numbers to be added.

The idea of a flow chart to give expression to an algorithm is inherent in this demonstration but it needs further discussion and direction when the teacher transfers this experience to real computers. A follow-up activity could be something on the order of the exercise shown below.

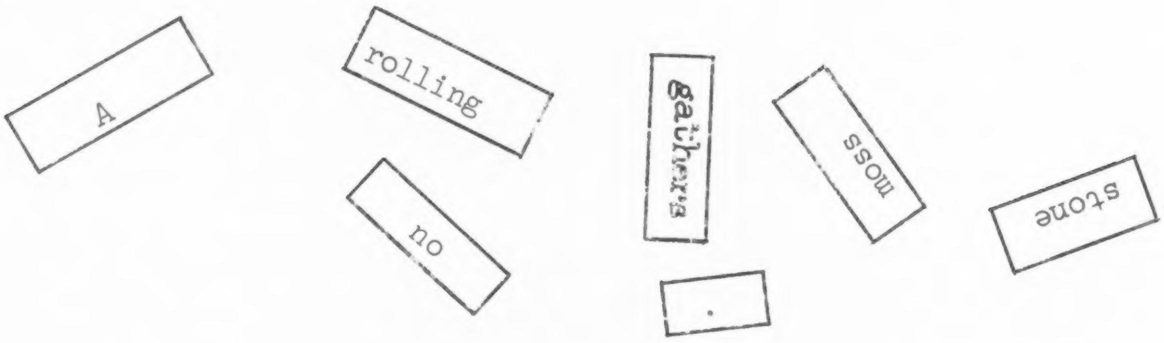
Complete the exercises using the sample as a guide.

Sample:

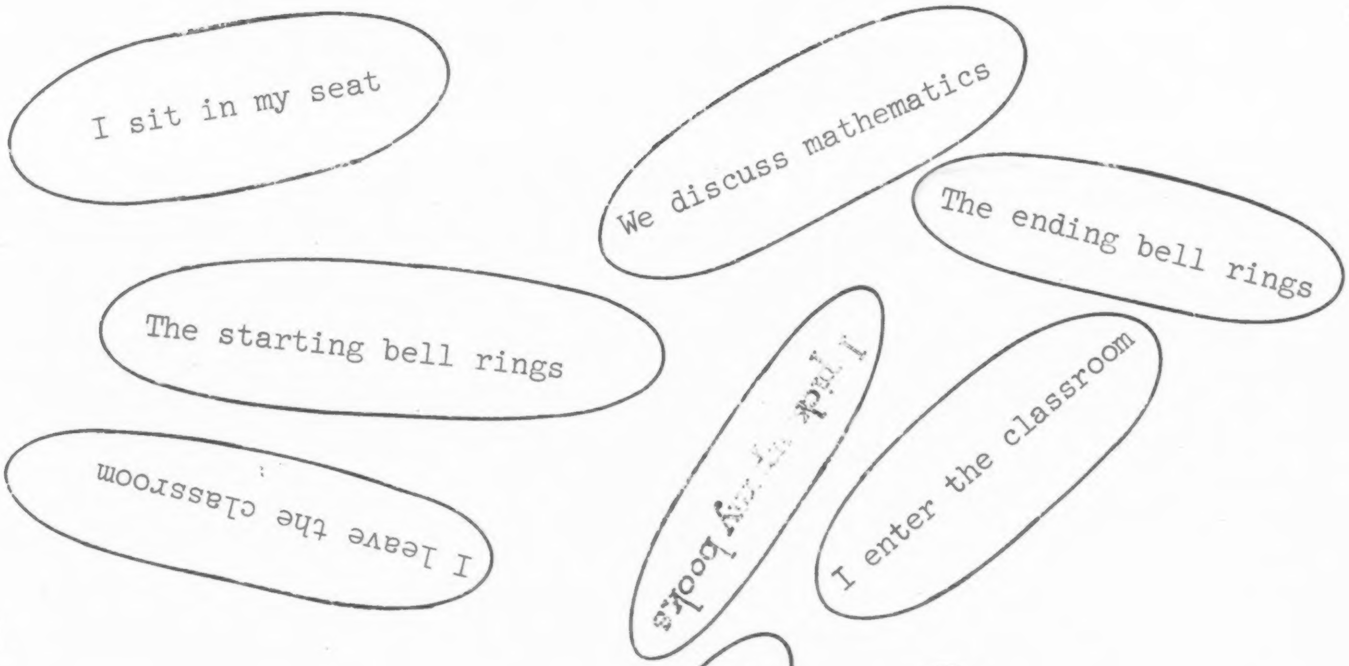


Exercises:

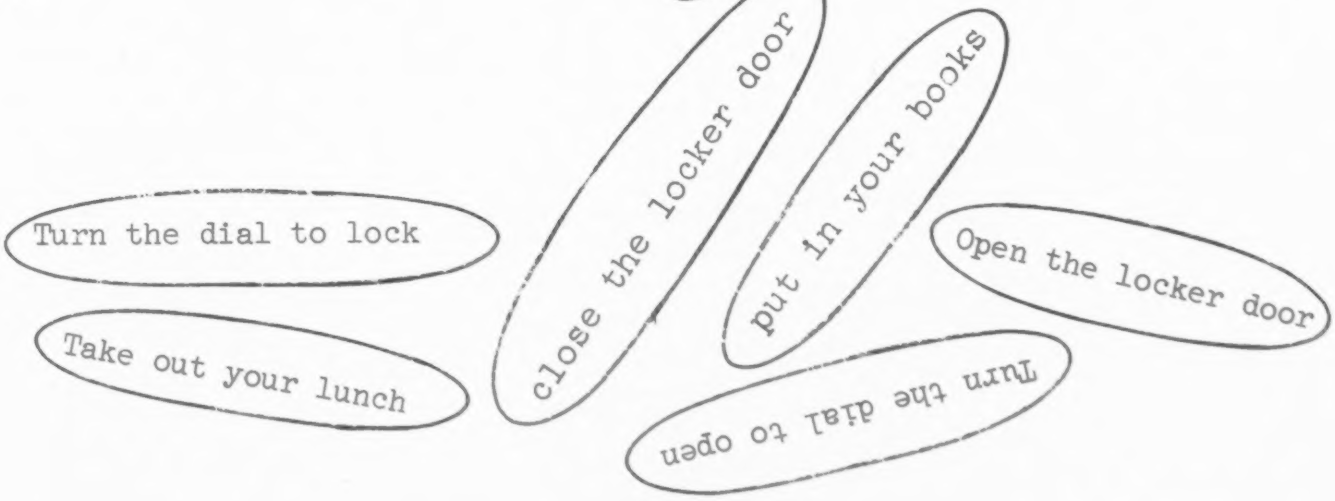
(a)


A
rolling
no
gathers
moss
stone
.

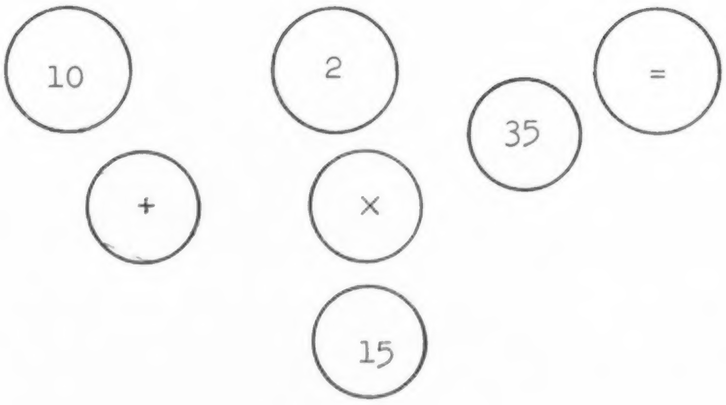
(b)


I sit in my seat
We discuss mathematics
The ending bell rings
The starting bell rings
I pick up my books
I enter the classroom
I leave the classroom

(c)

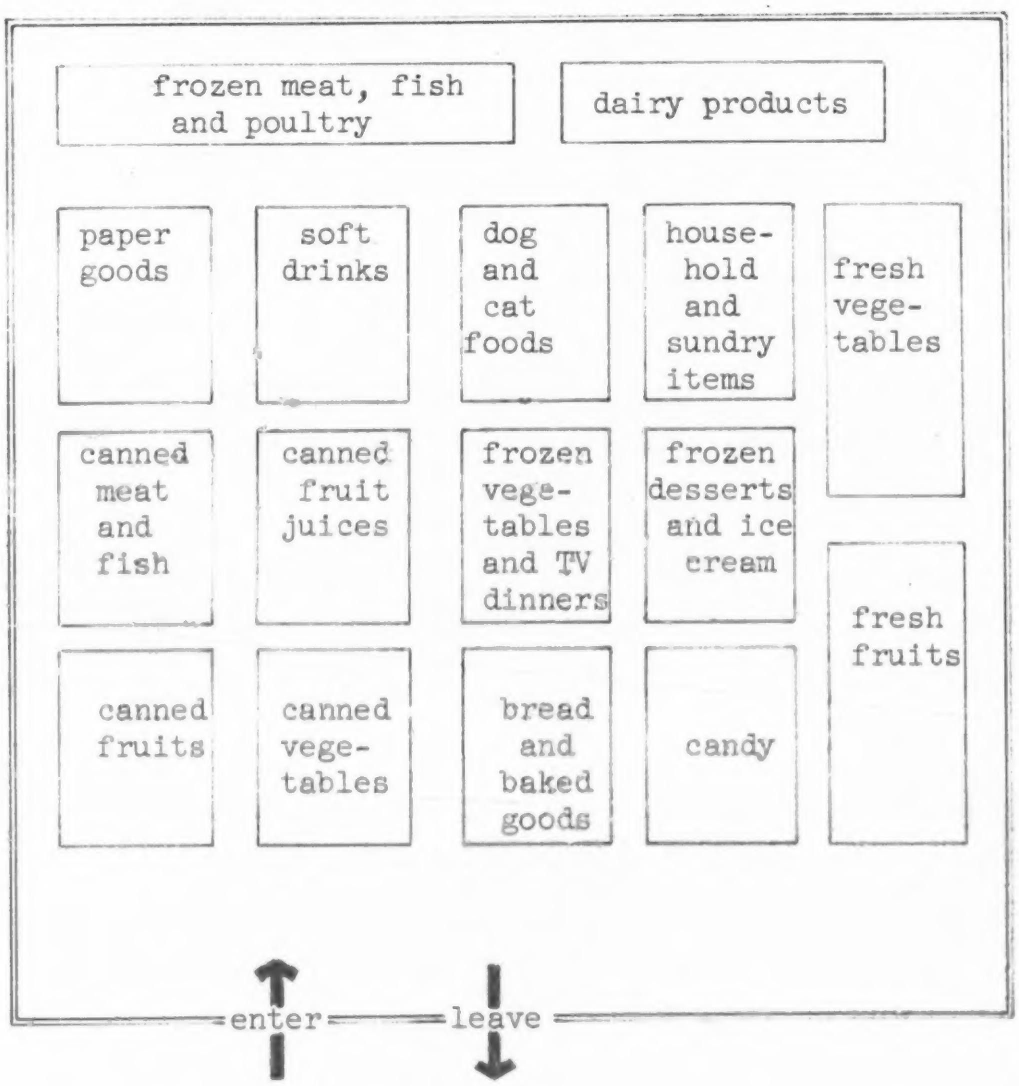

close the locker door
put in your books
Turn the dial to lock
Open the locker door
Take out your lunch
Turn the dial to open

(d)



The previous exercises were intended to show that there is at least one order that could be associated with the regions of each problem. The following exercise gives the student an opportunity to arrange a sequence for a list of actions based on a given situation.

Exercise: Often we go into a supermarket without any idea of how we are going to accumulate the items we wish to purchase and the result is that we end up walking back and forth, sometimes unnecessarily so. Arrange a tour through the store below, using arrows so that the shortest trip is made in order to pick up the items on the list.


frozen meat, fish and poultry
dairy products
paper goods
soft drinks
dog and cat foods
household and sundry items
fresh vegetables
canned meat and fish
canned fruit juices
frozen vegetables and TV dinners
frozen desserts and ice cream
fresh fruits
canned fruits
canned vegetables
bread and baked goods
candy
enter
leave

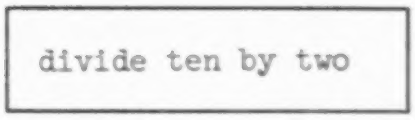
List

1 doz. eggs
2 cans of peas
4 T.V. dinners
2 lbs. fresh tomatoes
1 box face tissues
1 can opener
1 loaf of bread
1 carton diet-cola
$\frac{1}{2}$ doz. muffins

An expression of how to do something is essentially what is meant by a flow chart; however, it is expedient for pedagogic reasons as well as our desire to have this work computer oriented, to adopt a suitable flow chart notation. The introduction of this notation could take the following form:

The action box is for action to be taken. We will use a rectangle which contains a statement describing the action to be taken, e.g.,

set the clock



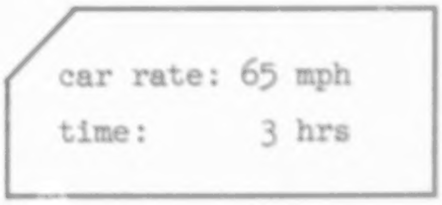
This could be followed by an exercise consisting of statements which imply action and some which do not. It would be the task of the student to select those statements which would be appropriate for use in an action box (assignment box is the term we wish to use eventually but it would be too confusing to use at this time), e.g.,

What time is it?	(no)
Close the door	(yes)
A brown house.	(no)
Add seven to three.	(yes)

Inasmuch as flow charting concerns itself with classes of problems and not a specific problem, it is questionable as to whether or not the following early examples should include constants, since they really never appear on a flow chart. However, the reason for employing a flow chart sequence of a problem using constants at this stage is to try to relate specific problems with which the child is already familiar to the idea of a class of problems.

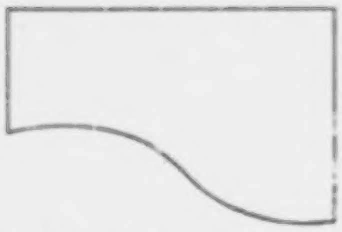
The input box is reserved for information that must be given in order to solve a problem, e.g.,

A car travels at an average rate of 65 mph for a period of 3 hours.



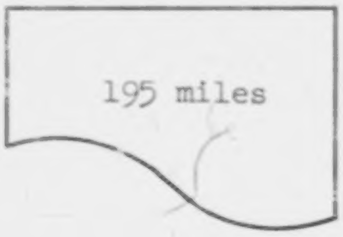
The left corner is cut to resemble the popular version of a punched card. (Ref.: Algorithms, Computation and Mathematics, SMSG, page 39)

The output box is represented by the form



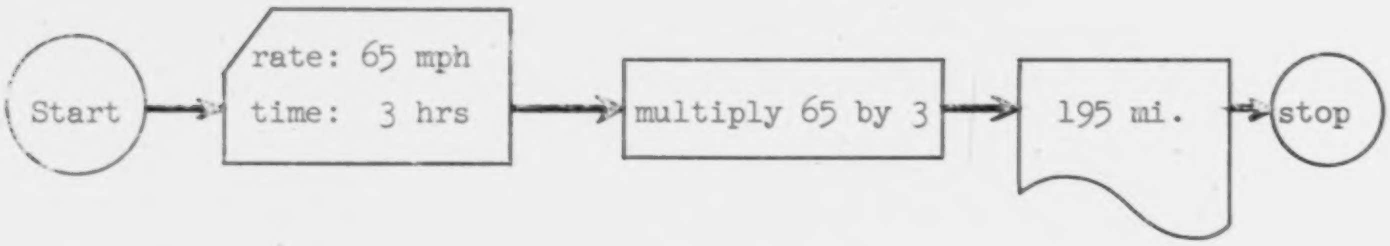
which suggests a piece of paper torn from a typewriter or line printer.

Suppose in the example just given you are to find the distance traveled for the given time; then the output box would have in it



which is the information resulting from the solution.

If we want to construct a flow chart for the same problem then we can use the boxes and arrows together to get



The circular start and stop boxes suggest the round buttons commonly used to start and stop pieces of machinery.

(a) Which is the output box?

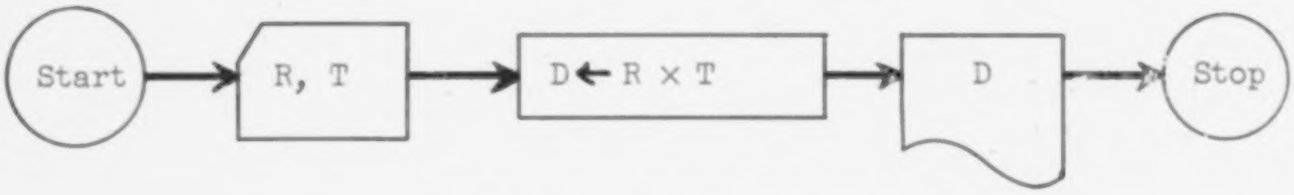
(b) Which is the input box?

(c) Which is the action box?

(d) Construct a flow chart for finding the distance a car travels if it's average rate of speed is 45 mph and the time in travel is 6 hours.

(e) How does this flow chart differ from the first one?

(f) Using R to represent the rate of speed, T to represent the time in travel, and D to represent the distance try to design a flow chart that could be used to express an entire class of problems of the same kind.



(It is not likely that the student will produce the chart illustrated here, since there are many ways of denoting what is to take place. After proper recognition of any creative result on the part of the student, it is suggested that a discussion lead by the teacher and inspired by the variety of attempts, be used to introduce the form above. The purpose is simply to provide consistency in form and notation, and to be expedient in our objective of learning how to use flow charts as an expression of an algorithm.)

The input box is representative of a set of punch cards each of which has some input data which will involve the constants necessary to perform a specific problem, but the flow chart being an expression of an algorithm is not intended to be concerned with specific problems and so variables rather than constants are used. Samples of variables as used in computer language are:

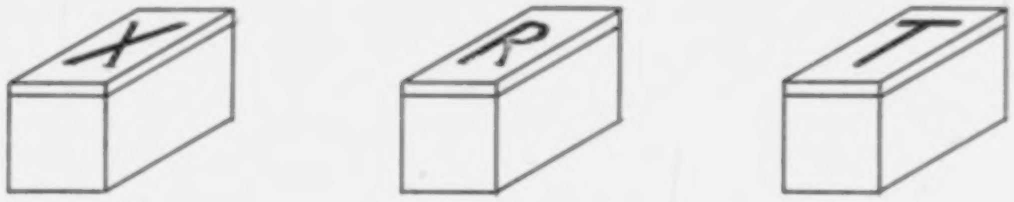
A, B, X, T, R, Y or such descriptive combinations of letters as: DIST, AREA, LENGTH, FLRR. (For a more detailed explanation of variables as used in computer language, see ACM Sec. 2-4.)

Notice that the variables are limited to upper case Roman letters which are the usual symbols available to computers. Also, there are not enough letters available for use as variables so combinations of letters are employed. Sometimes a descriptive combination of letters helps to remind us of how the variable is being used. We regard an unbroken string of letters (no intervening punctuation, operation symbols or parentheses) as one variable or one symbol. That is to say, an expression like XN is not considered to contain either of the variables X or N but rather to be a symbol in its own right. When used for flow charting, no variable should be considered to appear as part of another variable. Samples of acceptable and unacceptable symbols for variables could take the form of an exercise, e.g.,

(1) P (yes)
(2) $P \times Q$ (no)
(3) 125 (no)
(4) VARIABLE (yes)
(5) $\frac{A}{B}$ (no)
(6) $A + B$ (no)
(7) AB (yes)
(8) 6 (no)

"In any computing problem, there corresponds to each variable used in that problem a location in the computer's storage. By *assigning* a number to a variable we mean simply reading the number (*destructively*) into the storage location corresponding to that variable. When evaluating arithmetic expressions a variable is to be treated as a name for the number to be found in the corresponding storage location. The number in the corresponding storage location is referred to as the *value* (or *current value*) of the variable. During the course of a computation many different values (perhaps even millions) may be assigned to a given variable. Thus it will not be meaningful to speak of the value of a variable without specifying the time or, more precisely, the stage of the computing process. But once the stage of the process is specified, the value of the variable is uniquely determined.

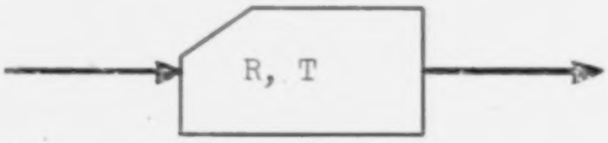
A storage location may be hard to visualize. If so, here is an analogy which cannot lead to error. Consider that to each variable there corresponds a wooden box. To make the correspondence clear we engrave on the boxes the corresponding variables. (But remember that the variable is a name not for the box but for the number inside.)



Three boxes with Identification

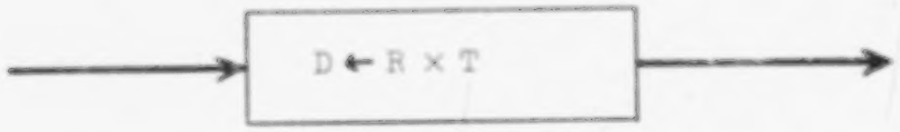
Now if we want to assign 2.5 to the variable X, we open the box labeled X, dump out the contents and put in 2.5."

Assignment may be done in an input step as in the previous flow chart example. When we come to the input box



we empty out the boxes labeled R and T and fill them respectively with the values punched on an input card.

Another important way of making an assignment is by means of the action box which hereafter will be referred to as an assignment box. In our previous flow chart example the assignment box looked like the following:



The product of R and T is assigned (indicated by a left-pointing arrow) to variable D. At this time we empty out the box labeled D and fill it with the product resulting from R × T.

Some examples of inadmissible and admissible assignment boxes will help the student to understand their limitations.

	Assignment Box	Admissible or not	Reason if not admissible
(a)	$2 \leftarrow 1 + 1$	(no)	Assignments are made only to variables, not to constants.
(b)	$A \leftarrow L \times W$	(yes)	
(c)	$R \times T \leftarrow D$	(no)	$R \times D$ is not a variable.
(d)	$D \leftarrow R \times T$	(yes)	
(e)	$P \leftarrow 2 \times (L + W)$	(yes)	
(f)	$A \leftarrow 3$	(yes)	
(g)	$4 \leftarrow 3$	(no)	Assignments are never made to constants.
(h)	$2 \times (L + W) \leftarrow 12$	(no)	$2 \times (L + W)$ is not a variable.
(i)	$A \leftarrow B$	(yes)	

Referring once more to the previous flow chart example,

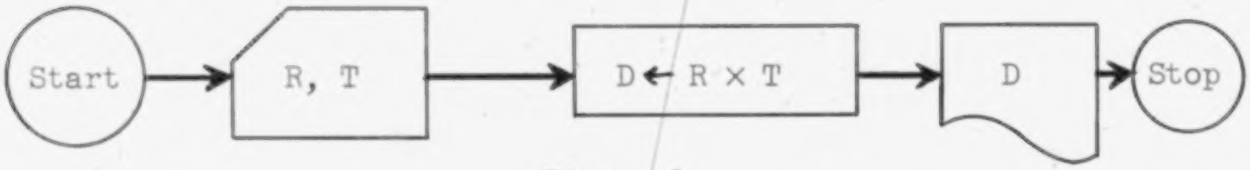


Figure 1

We have learned about three basic kinds of steps that occur in the sequence of a flow chart called "input", "assignment", and "output".

Step 2 is the __________ box (assignment)
Step 3 is the __________ box (output)
Step 1 is the __________ box (input)

Note: The problems shown below are based on some that appear in the SMSG Mathematics For The Elementary School Grade 6, Part II. This is a very elementary stage in flow charting and does little more than offer the student a plan by which he can visualize the procedural sequence for a given problem from start to finish. It does, however, provide exercise in the use of variables as developed in the earlier flow chart discussion. Perhaps this stage of the development would be most suitable with the proposed Grade 7, Chapter 5 on Measure.

Exercises.

In each of the following exercises your job is to convert the Instructor's problems into a flow chart similar to that of Figure 1.

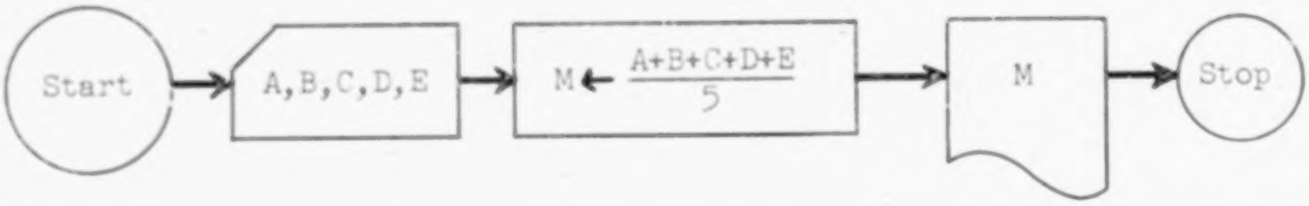
1. Mike went bicycle riding every day after school for a week and he kept a record of the distances he traveled. The distances were A, B, C, D, and E miles. What was the average distance traveled per day?

2. Terri divided Q quarts of ginger ale among F friends at her birthday party. How many ounces did each guest get? (There are 32 ounces in a quart.)

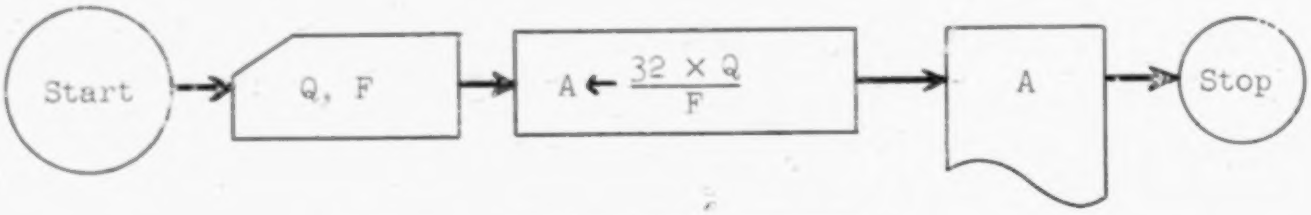
3. How many shoe boxes can be packed in a carton whose base is 2 sq. ft. if the carton is b ft. high? Each shoe box is $\frac{a}{2}$ ft. by $\frac{a}{3}$ ft. by $\frac{b}{3}$ ft.

Answers to Exercises.

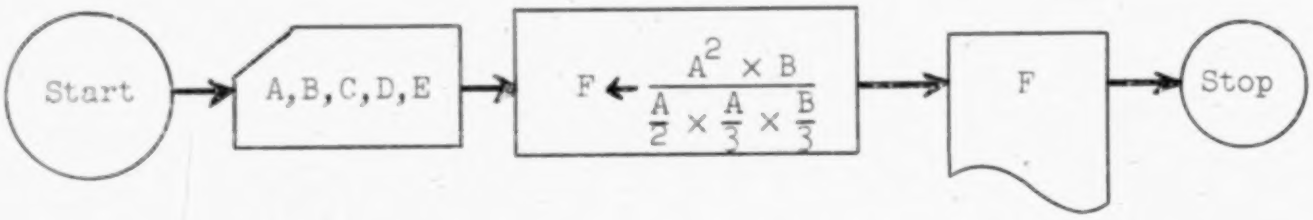
1.


Start
A,B,C,D,E
$M \leftarrow \frac{A+B+C+D+E}{5}$
M
Stop

2.


Start
Q, F
$A \leftarrow \frac{32 \times Q}{F}$
A
Stop

3.


Start
A,B,C,D,E
$F \leftarrow \frac{A^2 \times B}{\frac{A}{2} \times \frac{A}{3} \times \frac{B}{3}}$
F
Stop

If a flow chart is truly an expression for a class of problems then the same flow chart should be an effective representation for the solution of many specific problems of the same type. Consider the car problem again, only this time there are many situations which require a solution and the data for each of these is given in the table shown.

CAR PROBLEM DATA TABLE

R(mph)	T(hrs)
47.0	6.3
54.6	2.7
11.7	0.1
36.8	0.5
64.4	3.2
:	:
58.3	2.9

If we have a stack of input punch cards each containing a different single line of data for R and the corresponding T from the table, then this stack could represent the entire table. The only change needed to be made to the flow chart in Figure 1 is that it be repeated and simple repetition is easy to express in flow chart language, by forming a loop as shown in Figure 2.

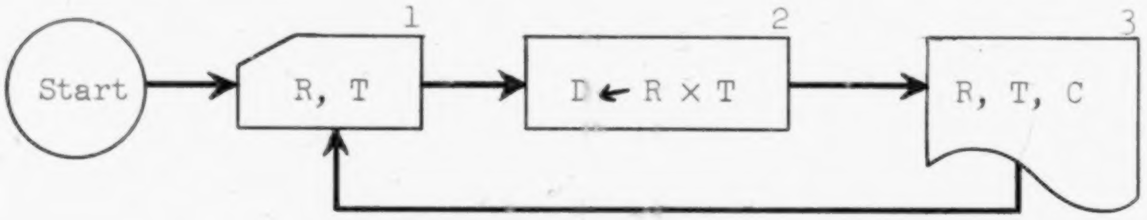


Figure 2

Now the flow chart tells us that after starting:

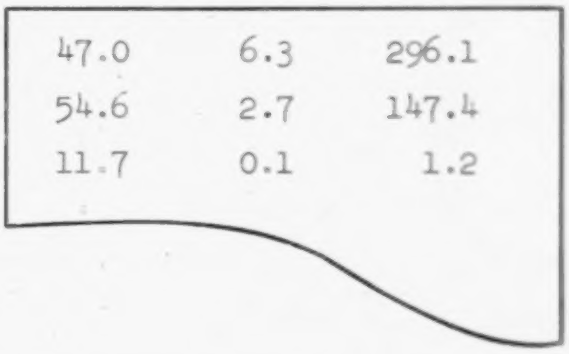
Step 1. Assign the data from the first punch card to the variables R and T in the input box. One may think of a messenger who takes the number 47.0 from the first punch card to the wooden box labeled R, dumps out what may be in it, and puts the 47.0 into it instead. He also puts 6.3 in the wooden box labeled T in the same manner. (This is what is meant by destructive read-in.)

Step 2. The product of R and T is assigned to D. This time our messenger empties the wooden box labeled D and replaces the contents with the number resulting from the product of R and T.

Step 3. This step merely calls for printing the current value of D (just computed in Step 2) along with the first punch card constants for R and T.

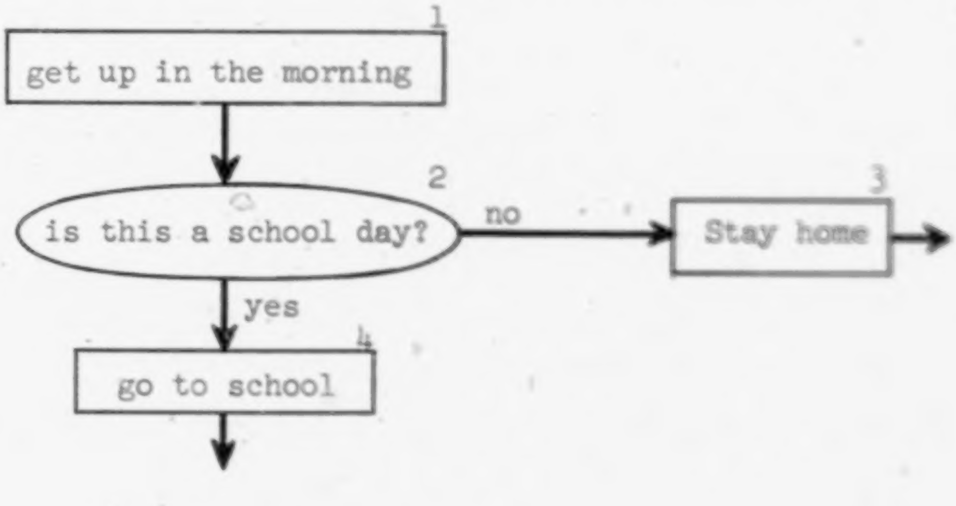
Instead of stopping the process after Step 3, the flow chart tells us (by the arrow leaving the output box) to go back to the input box, remove the first punch card from the stack, allowing the next card (if there is any) to become first, and repeat the process. It is understood that if a flow chart arrow carries us into an input box and there aren't any punch cards left in the stack, then the computation is to stop. Otherwise, Figure 2 would suggest an endless loop with no way of stopping.

Since by this repetitive process there is likely to be many printings (each to indicate a value of D), putting the variables R and T along with D in the output box of the flow chart arranges for the printing of the contributing data with each value of D. This enables a person to know just which specific problem the value of D is a solution to. A sample of the print-out for the first three punch cards is shown below.



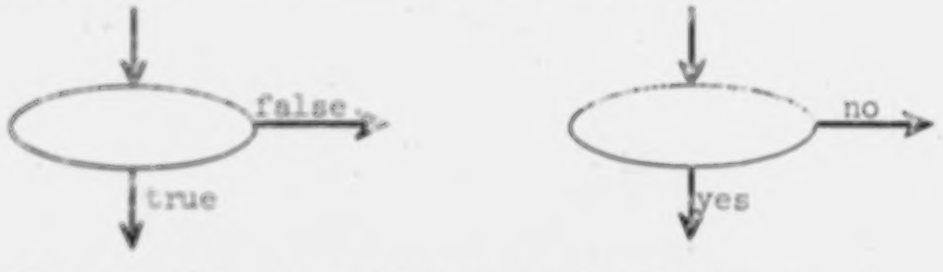
```
47.0     6.3    296.1
54.6     2.7    147.4
11.7     0.1      1.2
```

When a person attempts to solve a problem one technique in organizing his thoughts is to make a list of the things to be done which includes the order to be followed. There are times, however, when one approaches a fork in the road and what follows is a direct consequence of the decision made at the fork. For example,



Which way to go after Step 2 depends on the answer to the question in Step 2. If the answer is yes the next step is 4.

This new addition to our flow chart language is called a decision box or a condition box and will appear oval in shape.


false
true
no
yes

With this new flow chart tool we hasten to improve our car problem flow chart to that of Figure 3.

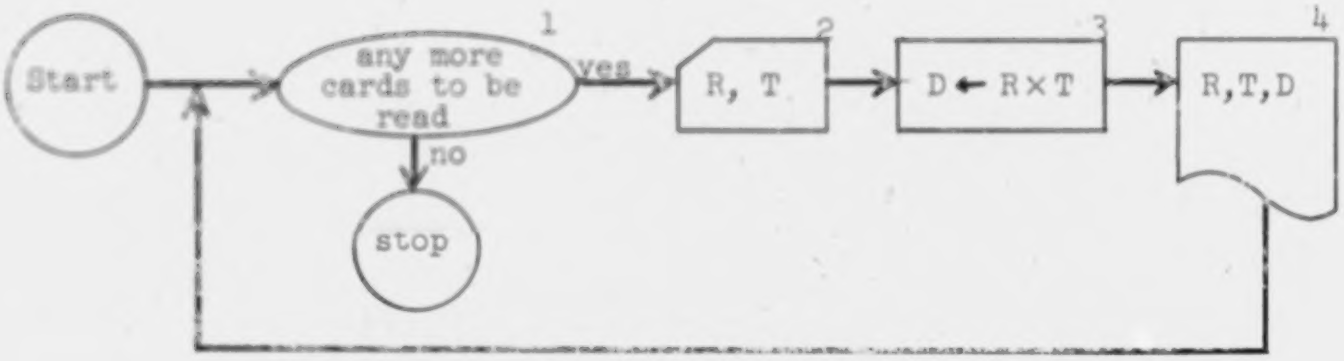
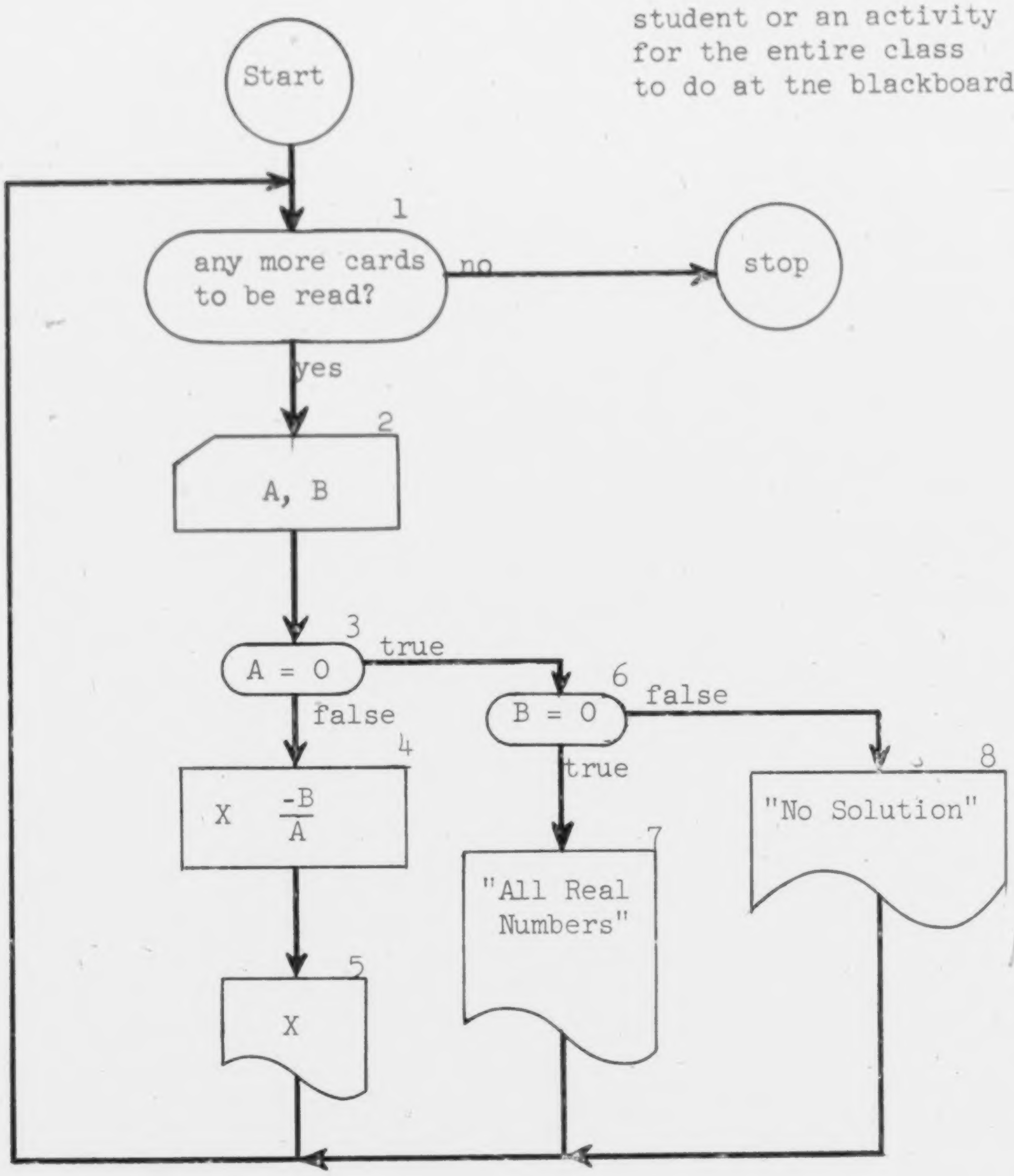

Start
1
any more cards to be read
yes
no
stop
2
R, T
3
D ← R×T
4
R,T,D

Figure 3

At this stage of our flow chart development we have accumulated some basic essentials adequate for expressing many mathematical algorithms.

$ax + b = 0$

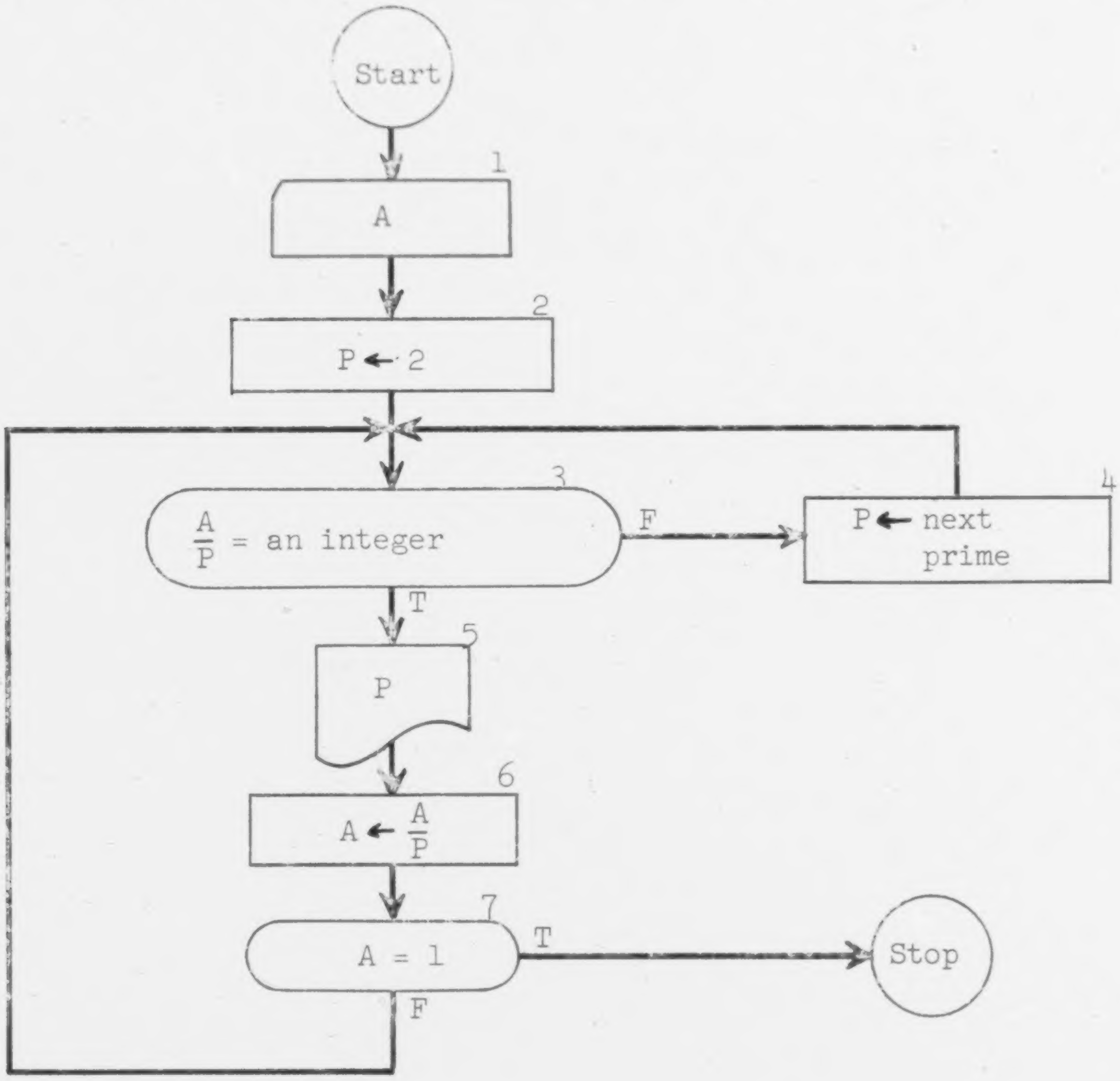
Suggested for the
proposed Grade 7,
Chapter 8, either as
an exercise for the
student or an activity
for the entire class
to do at the blackboard.


Start
1
any more cards to be read?
no
stop
yes
2
A, B
3
A = 0
true
false
4
X -B/A
5
X
6
B = 0
false
true
7
"All Real Numbers"
8
"No Solution"

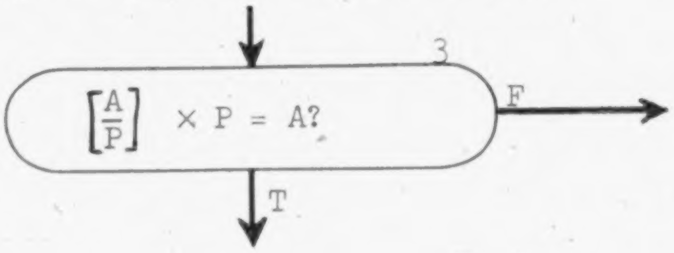
The prime factorization of whole number A, $A \neq 0$

The flow chart that follows is a suggestion that takes liberties with the more formal notation actually used in computer work. It is done in the interest of providing a clear picture of the mathematical algorithm involved with loss of the intricacies of a more complex flow chart. For example, it is assumed in the chart below that a list of primes is available. This is certainly a possibility and there are flow chart procedures for calling on these rather than saying "assign the next prime to P" in Step 4.

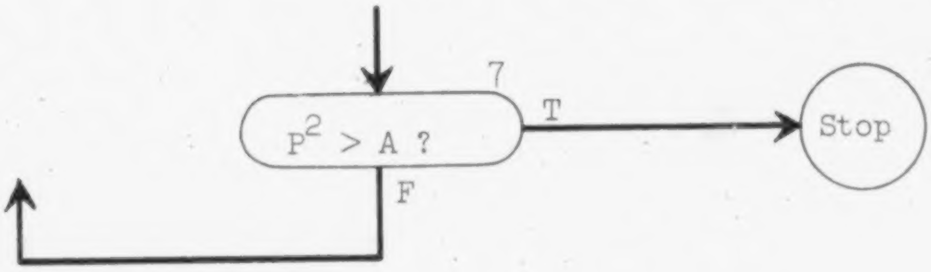
A likely place for this activity appears to be the proposed Chapter 5, Grade 8, on number theory.



Note: For box 3, the teacher may prefer to check for P as a divisor of A by using the greatest integer function


3
$\left[\frac{A}{P}\right] \times P = A?$
F
T

For box 7, the teacher may prefer to use


7
$P^2 > A$?
T
F
Stop

PROBABILITY AND STATISTICS

Richard Dean and Martha Zelinka

Put succinctly, the goal for the probability and statistics material in grades seven through nine is the capability of understanding, for example, the report of the Surgeon-General on smoking and cancer. The general attack is to present in grades seven and eight the material on probability written by SMSG, INTRODUCTION TO PROBABILITY, which is now being revised. The material for grade nine on statistics does not appear to have been written, and we give here only the briefest of outlines for it. Undoubtedly we are trying something new here, but we have in our new syllabus for seven through nine more power and a considerably more sophisticated student. Remember too that the slower student is expected to take longer to do all this material. The grade placement of all the material should be judged only for the college-capable student. No one has yet challenged the utility of statistics nor the social responsibility of mathematics to present it. What remains is the real challenge of writing it so that it becomes feasible for students at this level.

A number of "tie-ins" exist to connect probability and statistics with the envisaged seven-nine syllabus. In grade seven computation of probabilities gives rise to computation with positive rationals and inequalities between positive rationals. Computation of the probabilities of the form $P(A \cup B)$ and $P(A \cap B)$ even use a little set theory! The use of tree diagrams is a crude flow chart; perhaps a real flow chart could be drawn to solve a whole class of probability problems. Of course the entire subject is an open invitation to modeling, but basically the most often used tool is combinatorial counting. Please note that this is done in seven-nine without a tedious development of the calculus of permutations and combinations. Such a development is particularly true of the use of Pascal's triangle, which is done WST* as the attached sample shows.

---

*WST = "With Simple Tenderness", an instruction to the pianist from McDowel, "To a Waterfowl".

In grade eight, variance and standard deviation call for squares and square roots; a relocation of the origin to coincide with the mean uses translation of coordinates, change of scale, negative numbers, and absolute value. Also probability provides an opportunity to have another set function and to point out still one more "measure."

Nondiscrete probabilities can be finessed on a problem such as "What is the probability that when a stick is broken at two places, the three resulting parts form a triangle." This problem uses two-dimensional inequalities and further accentuates probability as area.

The notion of expected value arises with the measurement of any physical quantity and the inherent experimental errors.

General remarks: INTRODUCTION TO PROBABILITY was written without presuming the notion of "function." In rewriting, it will probably be expeditious to use this concept to obtain a simplified exposition. The same is true of absolute value, introduced in Chapter 10.

Special topics in Volume 2--Bernoulli trials, Bertrand's ballot problem, and Markov chains--can be reserved for grades ten through twelve. A similar remark holds for permutations, combinations, n!, and Sterling's formula.

Topics in Probability
Grades 7, 8, and 9

Introduction to Probability
Parts I and II, Student Text, SMSG, 1966

This text is programed, and therefore the number of pages used for any topic might be misleading. It is important also to check with revised version (currently being done).

Grade 7:	Chapters 1-6, approximately three weeks
Chap. 1	Fair and Unfair Games
Chap. 2	Finding Probabilities
Chap. 3	Counting Outcomes: Tree diagrams Pascal's triangle without binomial theorem
Chap. 4	Estimating Probabilities by Observation: organization of data leading to notion of average and expectation
Chap. 5	$P(A \cup B)$) (WST)
Chap. 6	$P(A \cap B)$)

Grade 8:	Part I, Chapter 7, approximately three weeks Part II, Chapters 8 and 10
Chap. 7	Dependent and Independent Events (Review $P(A \cup B)$, $P(A \cap B)$.)
Chap. 8	Conditional Probability Bayes' Theorem
Chap. 10	Expectation Variation, Standard Deviation Normal Distribution) (WST!!) Physical Observations)

Grade 9:	
Chap. 1	Organization of data--grouping, histograms Continuous model of discrete situation Computation--algorithms for mean, variation, for grouped data

Grade 9: (continued)

Chap. 2 Estimation of mean and variance
Confidence intervals for mean
Chebyshev's Inequality (WST)

Chap. 3 Hypothesis Testing (Null hypothesis: Quality Control Errors of first and second kind)

Chap. 4 Binomial Theorem
Normal Distribution
Central Limit Theorem (WST)

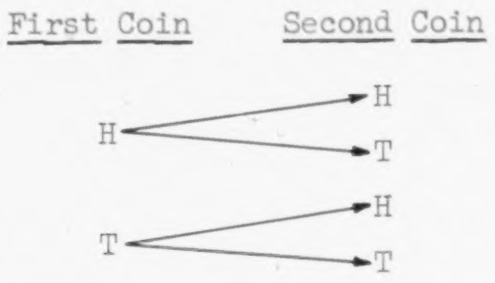
Pascal's Triangle

Introduce it as in Part I of Probability, Chapter 3, page 23. Do a little more than in Part I, Chapter 3.

List outcomes of tossing 1, 2, 3, and 4 coins. Make tree diagram, and list result in form of table. Continue as in Part 2, Chapter 9-3, page 207.

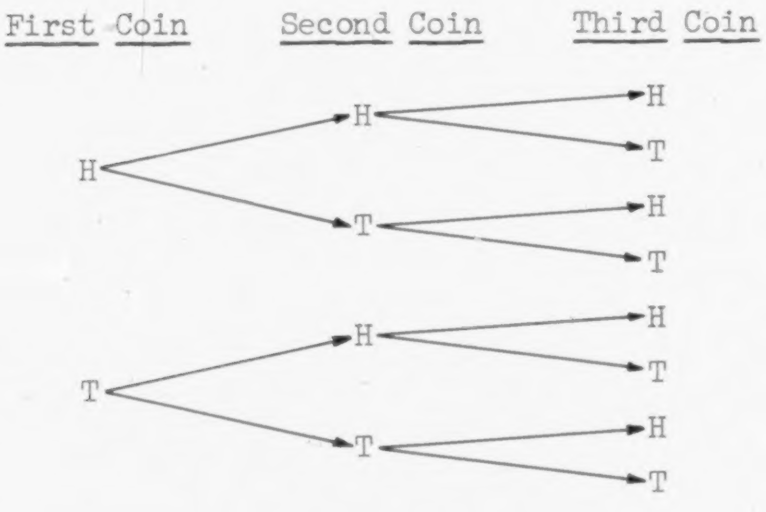
First Coin	Second Coin
H	H
H	T
T	H
T	T

As the number of outcomes increases, keeping track of the possible outcomes is more difficult. One useful way of listing them is by means of a "tree" diagram, as pictured below:

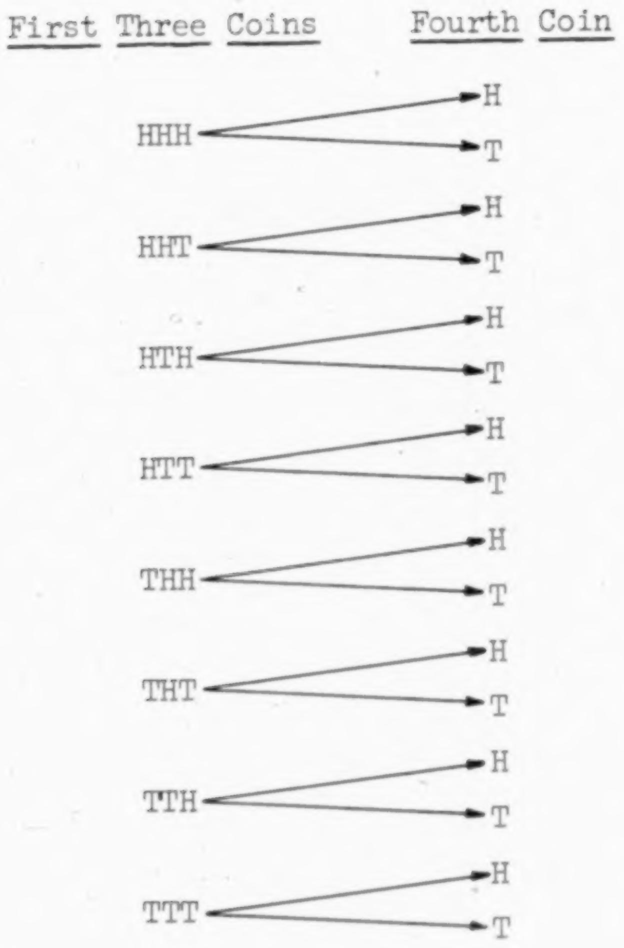

First Coin
Second Coin
H
T
H
T
H
T

Possible outcomes are HH, HT, TH, TT. There are four.

If a third coin is added, the number of possibilities is doubled again, as is seen in this diagram:


First Coin
Second Coin
Third Coin
H
T
H
T
H
T
H
T
H
T
H
T
H
T

Possible outcomes are HHH, HHT, HTH, HTT, THH, THT, TTH, TTT. There are eight.


First Three Coins
Fourth Coin
HHH
HHT
HTH
HTT
THH
THT
TTH
TTT
H
T

Possible outcomes are HHH, HHHT, HHTH, HHTT, etc. There are sixteen outcomes.

Now list the results in form of a table. How many times do 4, 3, 2, 1, 0 heads appear?

1 coin	1 head 1 time	0 heads 1 time			
2 coins	2 heads 1 time	1 head 2 times	0 heads 1 time		
3 coins	3 heads 1 time	2 heads 3 times	1 head 3 times	0 heads 1 time	
4 coins	4 heads 1 time	3 heads 4 times	2 heads 6 times	1 head 4 times	0 heads 1 time

Next write

1 1
1 2 1
1 3 3 1
1 4 6 4 1

and Pascal is born.

At this point, frequency distributions can be introduced aiming at recording properly data obtained in experiments.

— Look at Part II, Chapter 10-2, leading up to Expectation.

VECTORS ON A LINE

Hassler Whitney

Background:

Positive rationals; integers under addition; SMSG Grade 6.

Purpose:

(a) To show the theory of directed measurement, i.e., one dimensional vector space.

(b) The rational numbers operate; the operation on the rationals comes out as a corollary, in a natural manner.

Remarks:

This is an outline, showing a general method; hence some of it is rather sketchy, but should be easily filled in. Here are three principle examples:

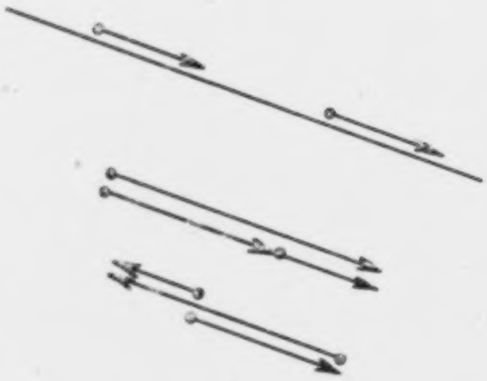
Example 1. An actual line, or line with origin; elements pictured as vectors (arrows). We tip the line, to hinder any concept of "the natural direction".

Example 2. Directed interval of time.

Example 3. The rationals, or reals.

The exposition is thought of as carried out for all these examples, at each point. Of course other examples may be brought in. We hope to end with some remarks on areas, for further understanding of both the general principles and applications.

1. Vectors.



We think of a line, and vectors, pictured by arrows, which may slide along the line. As a picture, each arrow has a start and an end.

Two vectors are added as follows: Usual way.

Associative law trivial. Commutative law: Interchange the two arrows. If one of them has the opposite direction, swing around a parallelogram for the picture.

By turning an arrow around, we form the opposite. Clearly

$$\text{opp opp } v = v;$$
$$v + \text{opp } v = \text{oppv} + v = 0.$$

(We note that we now have a group.)

Ex. For real numbers, opp 3 = $^-3$; opp $^-3$ = 3.

Ex. Time: If u is the time interval $t_2 = t_1$ (say 10 min.) and v is the time interval $t_3 - t_2$ (say -13 min.), then $u + v$ is the time interval $t_3 - t_1$ (say -3 min.).

Exercise. If Ann was 3 when Beth was born, and Beth was 2 when Carol was born, how old was Ann when Carol was born? (Ans: 5 or 6.)

2. Rationals as operators.

First define

$$0v = 0, \quad 1v = v, \quad 2v = v + v, \quad \text{etc.}$$

Just as the positive number line was extended, we now extend the above. This obviously gives:

$$-1v = \text{opp } v, \quad -2v = \text{opp } v + \text{opp } v, \text{ etc.}$$

Note that

$$\text{opp}v + \text{opp } v = \text{opp}(v + v).$$

More generally,

$$\text{opp } u + \text{opp } v = \text{opp}(u + v);$$

the above may now be written:

$$^{-}nv = n(\text{opp } v).$$

Also, turning the picture around (opp opp $v = v$),

$$^{-}n(\text{opp } v) = nv.$$

How did one picture $1/3$ on the number line? Go one third the way from 0 to 1. Do this with 0 and v, giving $(1/3)v$. Thus if this is v',

$$v' + v' + v' = v, \quad v' = (1/3)v.$$

We now picture $(2/3)v$: this is $2((1/3)v)$:

$$\frac{1}{3}v + \frac{1}{3}v = \frac{2}{3}v.$$

We now find $(m/n)v$. Just as for positive rationals,

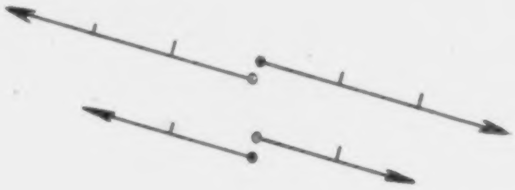
$$rv + sv = (r + s)v, \quad \text{positive rationals } r, s.$$

Again (picturing v as going from 0 to 1), the fact that $^{-}r = \text{opp } r$ makes clear that

$$rv + sv = (r + s)v, \quad \text{all rationals } r, s.$$

From the picture, it is clear that

$$-(2/3)v = \text{opp}((2/3)v) = (2/3)(\text{opp } v).$$



It is equally clear that

$$(2/3)v = \text{opp}(^{-}(2/3)v) = {}^{-}(2/3)(\text{opp } v),$$

which is the above with $2/3$ replaced by $^{-}(2/3)$. Thus we have:

$$(\text{opp } r)v = \text{opp}(rv) = r(\text{opp } v), \quad \text{any rational } r.$$

Apply this to opp v:

$$(\text{opp } r)(\text{opp } v) = rv.$$

Now (perhaps not before) apply this to the rationals:

$$\left.\begin{aligned}(\text{opp } r)s = \text{opp}(rs) = r(\text{opp } s),\\ (\text{opp } r)(\text{opp } s) = rs,\end{aligned}\right\} \text{ all rationals } r, s.$$

Give various numerical examples! Go through some proofs again, with numbers on number line in place of vectors in general.

It is also clear on a picture that

$$r(u + v) = ru + rv.$$

Next we note an obvious fact from picture: If $nu = nv$, then $u = v$. Converse clear. Hence also, obtain:

$$\text{If } ru = rv, \ r \text{ rational}, \ r \neq 0, \text{ then } u = v.$$

We still need the associative law. Take any v, and rational s. We have:

$$3[(\tfrac{1}{3}s)v] = (\tfrac{1}{3}s)v + (\tfrac{1}{3}s)v + (\tfrac{1}{3}s)v$$
$$= (\tfrac{1}{3}s + \tfrac{1}{3}s + \tfrac{1}{3}s)v = sv,$$
$$3[\tfrac{1}{3}(sv)] = \tfrac{1}{3}(sv) + \tfrac{1}{3}(sv) + \tfrac{1}{3}(sv) = sv = (\tfrac{1}{3} + \tfrac{1}{3} + \tfrac{1}{3})(sv)$$
$$= sv,$$

hence

$$(\tfrac{1}{3}s)v = \tfrac{1}{3}(sv);$$

more generally,

$$(\tfrac{1}{n}s)v = \tfrac{1}{n}(sv).$$

Also, we have

$$(\frac{1}{n}\,s)v + (\frac{1}{n}\,s)v + (\frac{1}{n}\,s)v = (\frac{1}{n}\,s + \frac{1}{n}\,s + \frac{1}{n}\,s)v$$

$$= [(\frac{1}{n} + \frac{1}{n} + \frac{1}{n})s]v = (\frac{3}{n}\,s)v,$$

$$\frac{1}{n}(sv) + \frac{1}{n}(sv) + \frac{1}{n}(sv) = (\frac{1}{n} + \frac{1}{n} + \frac{1}{n})(sv) = \frac{3}{n}(sv);$$

using what was proved above gives, replacing 3 by m,

$$(rs)v = r(sv),$$

where at the moment r must be a positive rational. But if r is negative, then opp r is positive, and we have x writing r' = opp r,

$$(rs)v = ((\text{opp } r')s)v = (\text{opp}(r's))v = \text{opp}((r's)v) = \text{opp}(r'(sv))$$

etc. Hence the above holds for all rational r and s.

Applying this to the number line gives the associative law for multiplication.

Note that we have proved that our vector line is a vector space; or at least when we have proved that the real numbers form a field. For this, we must still show that multiplication is commutative.

As a little earlier, we find that

$$n(\frac{1}{n}\,s) = \frac{1}{n}\,s + \dots + \frac{1}{n}\,s = (\frac{1}{n} + \dots + \frac{1}{n})s = 1s = s,$$

$$n(s \cdot \frac{1}{n}) = s \cdot \frac{1}{n} + \dots + s\ \ \frac{1}{n} = s(\frac{1}{n} + \dots + \frac{1}{n}) = s \cdot 1 = s;$$

hence

$$\frac{1}{n}\,s = s \cdot \frac{1}{n}\,.$$

Continuing,

$$m(\frac{1}{n}\,s) = \frac{m}{n}\,s, \quad m(s \cdot \frac{1}{n}) = s \cdot \frac{m}{n}\ ;$$

hence

$$rs = sr,$$

at the moment if r is positive. For r negative, apply opposites as before. The required properties of our vector space and rational number system are proved.

3. Subtraction, multiplication, division.

We study these operations in the rational number system.

How do we solve

$$3 + ? = 7?$$

By examination, we see that 4 is the answer. Here is another way: From 3, we wish to get to 7. Just go back to 0, then go to 7:

$$3 + (^{-}3 + 7) = 7;$$

our new answer is $^{-}3 + 7$.

This seems foolish, for three reasons: It is complicated; it takes us through a much longer journey (is this really the same as 4?), and it requires using negative numbers, not needed in the answer. However, our symbols are supposed to be simply names for things; $^{-}3 + 7$ is another name for 4, and just our picture of it looks like a journey.

The new method is helpful when we wish to tell a person how to get from one number to another, i.e., solve

$$r + x = s,$$

in some fashion: the answer can be written as

$$x = \text{opp } r + s.$$

Suppose we give two definitions of the minus sign:

$$-r = \text{opp } r; \quad s - r = s + (-r).$$

These do not conflict, and the second reduces to ordinary subtraction; for $a - b = c$ if and only if $a = c + b$.

2

Various common properties are immediately verified. Thus,

$$-0 = 0; \qquad -(-r) = r;$$

$$-(r + s) = -r + (-s) = -r - s;$$

$$-(r - s) = -r + s = s - r;$$

$$(x + z) - (y + z) = x - y;$$

$$(x - z) - (y - z) = x - y.$$

(The last two will be compared with formulas for multliplication and division.)

We know already that

$$(-x)y = x(-y) = -xy,$$

$$(-x)(-y) = xy.$$

Two more easy formulas:

$$x(y - z) = xy - xz, \quad (-x)(y - z) = x(z - y) = xz = xy.$$

Others now follow.

Theorem. In a vector space, $rv = 0$ if and only if $r = 0$ or $v = 0$.

We know that $0v = 0$ (two senses of "0"); $r0 = 0$, at once from definition. If $v \neq 0$ and $r \neq 0$, the definition of rv shows that $rv \neq 0$ (use $(1/n)v$, then $(m/n)v$.) For the rationals, the usual theorem on $xy = 0$.

For ease in studying division, we wish something like "opposite" which works for multiplication in place of addition. What replaces 0? Its property for addition is: $x + 0 = 0 + x = x$, all x. For multiplication, 1 has this property: $1 \cdot x = x \cdot 1, = x$, all x.

Now in place of "opposite" we have "reciprocal":

Theorem. If $x \neq 0$, there is a unique number y such that $xy = 1$.

If $x = m/n$, use $y = n/m$; proof immediate, as in Grade 6. In fact, see the diagram for a special case: $x = 3/5$; a third of x is $1/5$, and five thirds of x is $5/5 = 1$. For negatives, say $x = -3/5$, use $y = -5/3$.

Now find the unique solution of

$$ax = b \quad (\text{where } a \neq 0):$$

from a, go back to 1, then to b: Use $x = (\text{rec } a) \cdot b$:

$$a \cdot ((\text{rec } a) \cdot b) = (a \cdot (\text{rec } a)) \cdot b = 1 \cdot b = b.$$

Just as we introduced the minus sign, now introduce fractions:

$$\text{For } x \neq 0, \ 1/x = \text{rec } x, \quad \text{and} \quad y/x = y(\text{rec } x).$$

Now

$$x \cdot (1/x) = (1/x) \cdot x = 1,$$

$$1/x = y \text{ if and only if } xy = 1.$$

As for a formula for subtraction, we have

$$\frac{1}{xy} = \frac{1}{x} \cdot \frac{1}{y} .$$

Two formulas for subtraction give, for division:

$$\frac{1}{\frac{x}{y}} = \frac{y}{x} , \quad \frac{zx}{zy} = \frac{x}{y} .$$

The following are very easily proved:

$$\frac{a}{1} = a; \ \frac{0}{b} = 0 \ (b \neq 0); \ \frac{a}{b} \neq 0 \text{ if } a \neq 0 \ (b \neq 0).$$

Now

$$\frac{a}{b} = \frac{ad}{bd} , \ \frac{c}{d} = \frac{bc}{bd} \ (b , d \neq 0).$$

$$\frac{a}{b} = \frac{c}{d} \text{ if and only if } ad \cdot bc \ (b , d \neq 0).$$

Two formulas we specifically mentioned for subtraction give, (same method) for division,

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}, \qquad (b,d \neq 0),$$

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc} \qquad (b,c,d \neq 0).$$

The rules for addition are easily proved:

$$\frac{a}{c} + \frac{b}{c} = \frac{a + b}{c}, \quad \frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}.$$

As using opposites, using reciprocals correspondingly give

$$\frac{-a}{b} = \frac{a}{-b} = -\frac{a}{b}.$$

Now we find at once

$$\frac{a}{b} - \frac{c}{d} = \frac{ad - bc}{bd}.$$

Note that in all these, the symbols may represent any rational numbers. (Look at particular cases, integral and non-integral.)

All the usual working with fractions follows with ease from the above.

4. Directed measurement.

We suppose we have a set of "quantities", which form in a natural way a vector space.

Example 1. Directed line segments, parallel to a given line.

Example 2. Directed time intervals.

Example 3. Possible changes in quantity of water in a reservoir.

We know then how to add quantities, and multiply them by rational (or real) numbers.

Here is a particular question: In the given vector space, suppose we choose a fixed non-zero vector, say V, and wish to compare quantities (vectors) with this one. How do we do this?

For any vector v, we may write $v = aV$ for some number a. This gives a correspondence

$$F_V : v \rightarrow a; \quad \text{thus} \quad F_V(v) = a,$$

associating with the vector v the number a. If we call V the "unit", then we may say v "has a units"; this means merely that $v = aV$.

We can choose various "unit" vectors V; each gives an isomorphism of the vector space onto the vector space of real numbers.

Suppose we had the unit V, and now choose another, W. Then write

$$W = cV, \quad V = dW; \quad d = \frac{1}{c} .$$

Now for any vector v, if

$$v = aV, \quad v = bW,$$

then

$$v = a(dW) = (ad)W; \quad b = ad;$$
$$v = b(cV) = (bc)V; \quad a = bc.$$

However, it is much simpler to work directly in the vector space, and not in the number system, since the isomorphism depends on the choice of unit.

Example. With directed distances, ft and in are vectors, and 1 ft = 12 inches. This is a real equality. Length ℓ is 3 ft for instance.

OUTLINE -- VECTORS

H. S. Moredock
W. H. Sandmann

Part I -- Developing a Mathematical System

I. Scalars

Many of the quantities which we encounter can be described by a number (or measure) and a unit of measure.

Examples:

Length of an object - 5 inches
Outside temperature - 64 degrees
Mass of an object - 315 grams
Volume of a tank - 11 cubic feet
Speed of an airplane - 530 miles per hour
Time to eat lunch - 35 minutes
Distance between two cities - 118 miles

etc. (Ex. from physics - mass, energy, change - metric units)

In each case the measurement can be represented by a distance (interval) on an appropriate scale. These are numbers obtained from scale readings.

For these reasons we call these numbers scalars.

Operations with scalars - (real numbers)

Problems that call for adding, subtraction, multiplication, division.

(Review properties of an ordered field.)

Have students actually combine volumes, masses, etc. to fix operational properties of scalars.

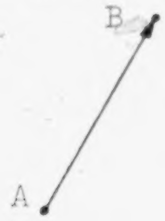
II. Vector Quantities

There are quantities that cannot be adequately described by a measurement on a scale alone.

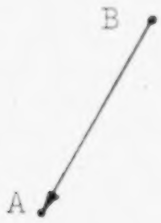
A. Describing trips.

If we wish to know the distance between two cities, we do not care to know particularly if the distance is measured from city A to city B or from city B to city A. We are interested only in a scalar, 115 miles.

However, if we are making a trip, it would make a difference to go from city A to city B, or to go from city B to city A. We use arrows to indicate these trips.



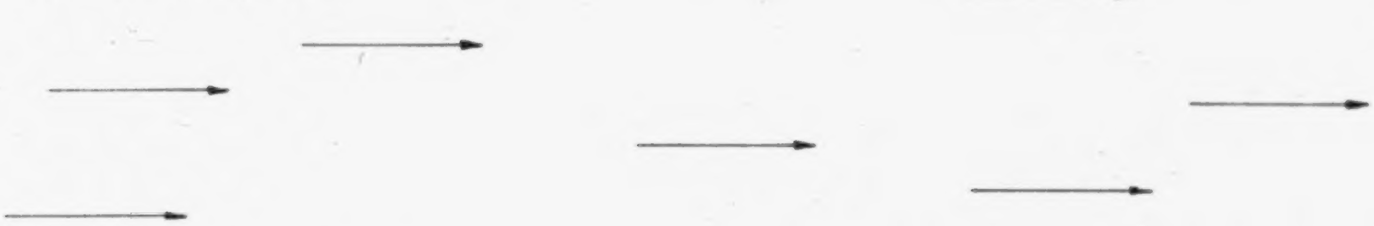
From A to B.



From B to A

Trips of two miles east along a road:

At various places:



Notice that these arrows convey two pieces of information. Their length represents the distance traveled, and the points of their arrows indicate direction. Each arrow has a starting point and an ending point.

B. Developing Meaning of a Vector.

It would be impossible to represent all the trips of 2 miles east. We would like to generalize all 2-mile trips east by a singe representation. In other words, we would like to form a mathematical model of all physical trips which are 2 miles east.

Place a piece of acetate over the arrows.

On the acetate, mark the starting point of each arrow.

Select one arrow. Slide the acetate (without rotating) so that the mark for the starting point of this arrow moves to the ending point.

Look at the other arrows. Are the marks on the acetate for their starting points now at the ending points?

Then the movement of the acetate, using one of the arrows as a guide, provides a representation for the various 2-mile east trips shown.

Start over again. This time mark some other possible starting points on your paper for 2-mile east trips.

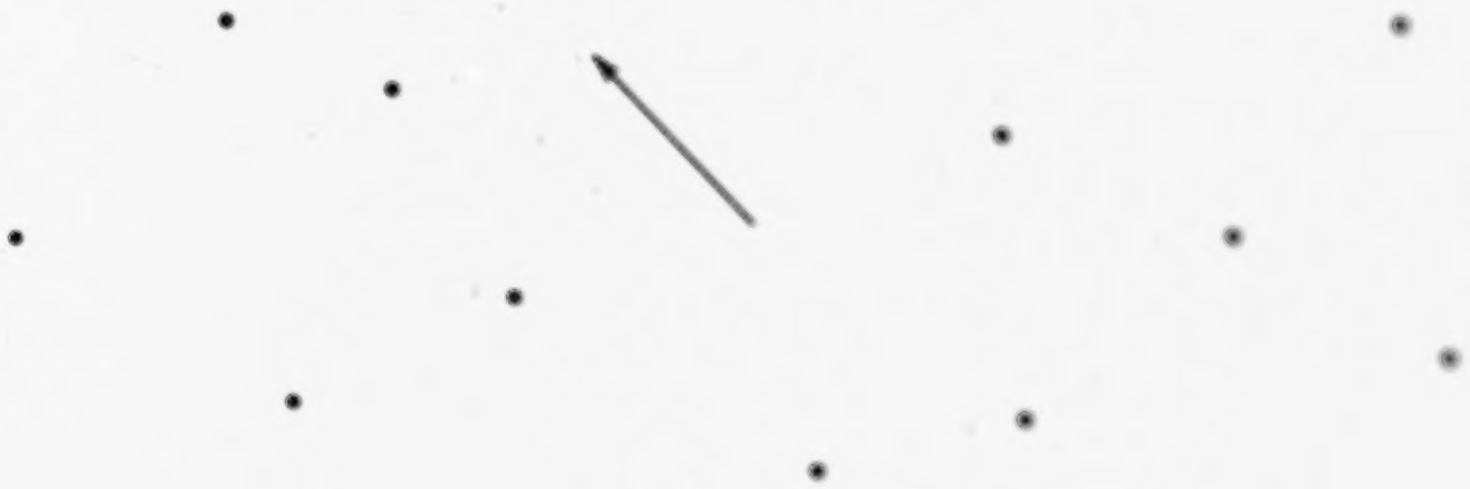
Place the acetate in the same beginning position as before, Mark on it the additional starting points you just made.

Using one of the arrows as a guide move the acetate without rotating so that the mark for the starting point now coincides with the ending point of the arrow.

Look at the position of the new marks. They show the ending points for the new arrows. (might make indentations for these ending points through the acetate onto the paper so that the new arrows can be drawn.)

This shows the generality of the representation. This on movement of the acetate, using a single arrow as a guide, represents all the 2-mile east trips by showing the ending point for any given starting point.

Go on to other examples, This time on paper show only one arrow representing a trip, and some scattered starting point for trips of the same type.



Place acetate on arrow and points, and mark the starting points on acetate, Move acetate as indicated by the one arrow. The marks on the acetate now show the ending points. From a few of the starting points draw arrows to show that these are trips of the same type -- i.e., same distance and direction.

Do enough of these to show that only one arrow is needed to provide the instructions for moving the acetate.

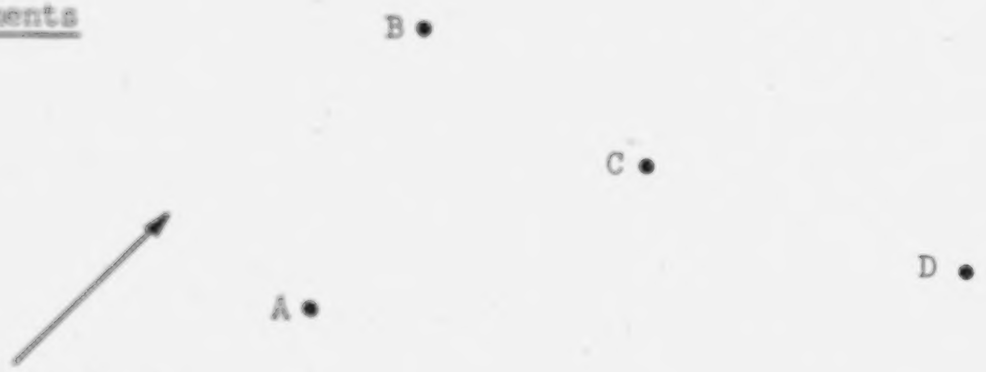
This gives an enlarged meaning for an arrow, from that of representing a single trip. As a "programmer" for moving the acetate, this arrow becomes a representative of a type of trip.

With this broader meaning, the arrow can now represent our idea of a vector.

At this time, the phrase "type of trip" can be replaced by the term "displacement."

Use the new terminology to develop familiarity.

III. Displacements



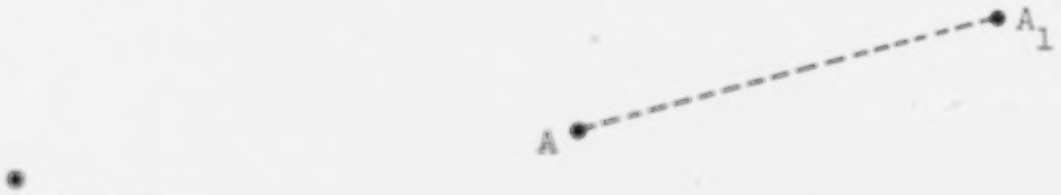
Displacement a

Given displacement a as defined. Apply this displacement to points A, B, C, C.

Lay on the acetate and mark the starting point of the arrow and the points. Move the acetate according to the arrow representing the displacement. Mark the ending points on paper and label them A_1, B_1, C_1, and D_1 respectively. A_1 is the result of applying displacement a to A, etc.

Do the inverse also. Given the application of a displacement to some points. Find the displacement.

Given: the application of a displacement to point A results in A_1. Find and represent the displacement.



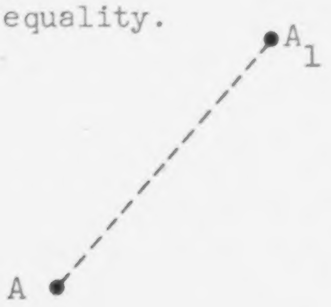
Displacement

Lay on acetate and mark point A and starting point where arrow representation is to be made, Slide acetate to where mark coincides with A. Mark ending point for arrow. Then draw the arrow representing the displacement.

Might consider extending displacement to objects and give representations. Have student move an object (block) on the blackboard or a chair in the room and have students represent the displacement on paper by a vector.

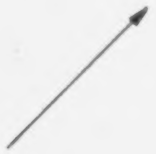
Keeping separate the representation of a displacement and the representation of its applications helps to keep these two ideas separate for the youngsters.

Develop definition for equality.

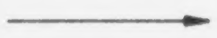


Suppose displacement a, when applied to A results in A_1, and suppose when displacement b is applied to A, A_1 is also the result.

A. Combining two displacements.

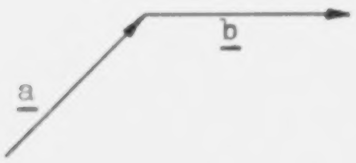


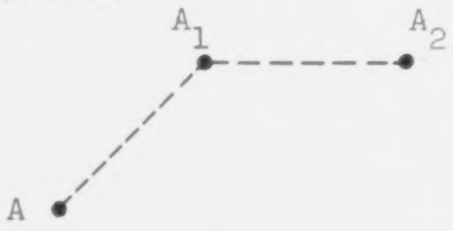
Displacement a



Displacement b

Given displacement a and displacement b as represented above. What is the effect of combining these two displacements on point A below? By "Combining" we mean applying one displacement and then the other. Observe the effect of applying to point A displacement a followed by b. We shall call this combining, a + b, and represent it as shown.

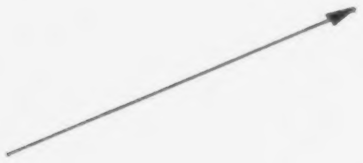


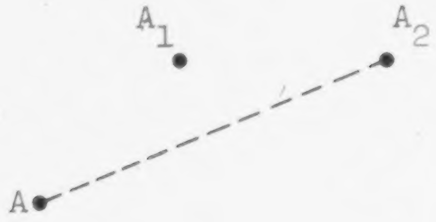


Displacement a + b
(displacement a followed by displacement b

Place acetate on arrow for a and point A. Move acetate according to arrow for a. Mark point A_1 on paper. Now place acetate on arrow for b and point A_1. Move acetate according to arrow for b. Mark A_2 on paper. The location of various points should be spotted on acetate. Point A_2 is the result of applying displacement a followed by b to point A.

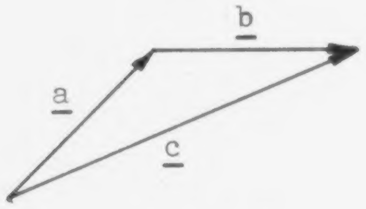
What would be the displacement that results in A_2 directly when applied to A? Let us call it $\underline{c}$.





Displacement $\underline{c}$

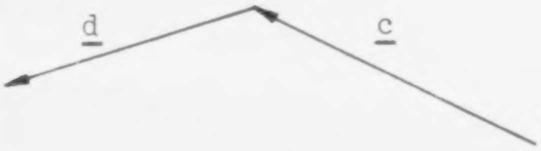
Since displacement $\underline{a+b}$ and displacement $\underline{c}$ have the same result when applied to point A, we write: $\underline{a+b} = \underline{c}$. This is summarized in the vector diagram



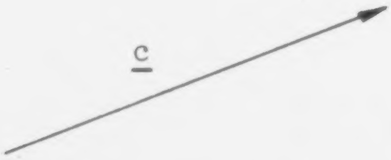
$\underline{a} + \underline{b} = \underline{c}$

(note: might introduce "resultant" here. If this is done it should remain with the physical model.)

Give many problems of this type:

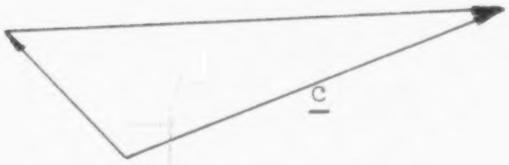


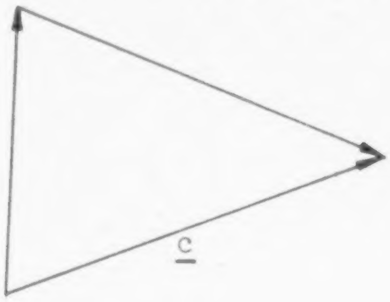
Given: Displacement $\underline{c+d}$, a combination of two displacements. Find a single displacement $\underline{e}$ which gives the same result. Relate these representations to displacements of objects. **Now go on to this type of problem.**



Given: displacement $\underline{c}$. Find two displacements $\underline{a}$ and $\underline{b}$ which when combined give the same result as displacement $\underline{c}$. (note: relate this to wanting to displace an object to a certain point but there are obstacles in teh way. Find a combination of displacements.)

Students will find that there are many correct answers for this problem.



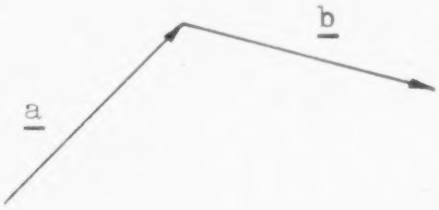


Show analogy with numbers. Given a number, 17, there are several pairs of numbers which have 17 as sum; 5, 12; 8, 9; etc. But for any pair of numbers there is a unique sum. Is this true of vectors?

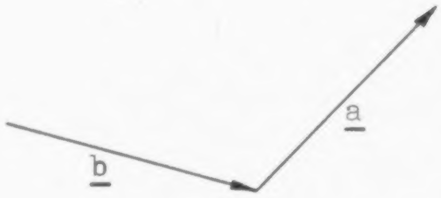
Discuss: closure property for vector addition. This helps in physics. Student needs closure in later work where closure mathematically has correspondance to physical world.

B. Commutative Principle.

Investigate another property. Given: two displacements a and b, combined in different order.

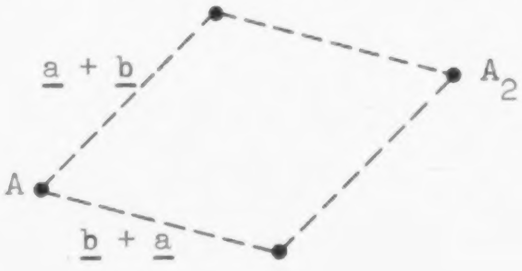


Displacement a + b



Displacement b + a

Apply these two displacements to a point, A. (Use acetate in usual fashion.)



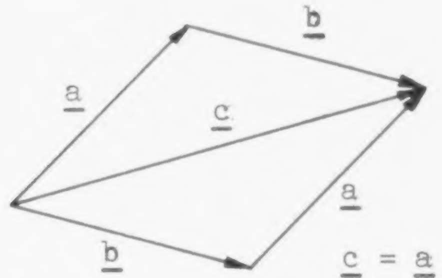
Do you get the same result?

Have students try come examples on their own before any generalizations are made.

Since displacement $\underline{a+b}$ and displacement $\underline{b+a}$ have the same result when applied to point A, we write,

$$\underline{a} + \underline{b} = \underline{b} + \underline{a}$$

This is a commutative property for vector addition. This is summarized in the vector diagram



$\underline{c} = \underline{a} + \underline{b} = \underline{b} + \underline{a}$

where displacement $\underline{c}$ is the single displacement which has the same result as either combination of displacements $\underline{a}$ and $\underline{b}$

Give serveral other exercises of this type.

C. Associative Principle

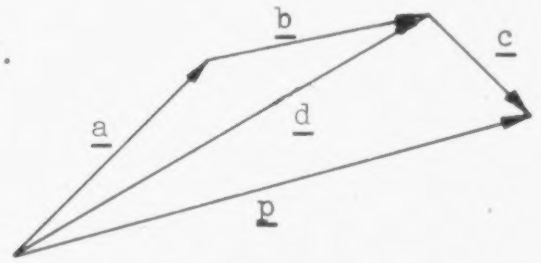
Combining three displacements. Given: three displacements $\underline{a}$, $\underline{b}$, and $\underline{c}$. Fine the combination, $\underline{a} + \underline{b} + \underline{c}$. Since displacements are combined in pairs, there are two ways of finding the combination, keeping the same left-to-right order of $\underline{a} + \underline{b} + \underline{c}$.

I. Find $\underline{a} + \underline{b}$, then find $(\underline{a} + \underline{b}) + \underline{c}$

II. Find $(\underline{b} + \underline{c})$ then find $\underline{a} = (\underline{b} + \underline{c})$

Let us look at the vector diagram for each way.

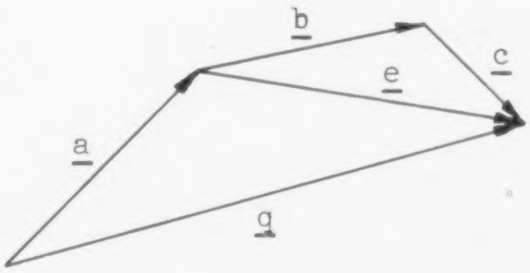
I.



$(\underline{a} + \underline{b}) + \underline{c}$

Note that $\underline{d}$ is found as the single displacement for $\underline{a+b}$ then $\underline{p}$ is the single displacement for $\underline{d+c}$ or $(\underline{a} + \underline{b}) + \underline{c}$

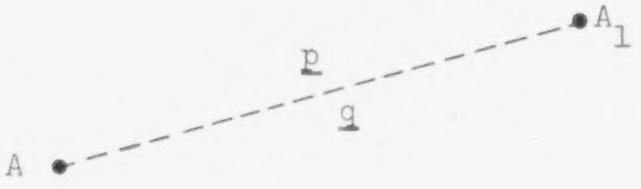
II.



$\underline{a} + (\underline{b} + \underline{c})$

Note that $\underline{e}$ is found as the single displacement for $\underline{b+c}$, so $\underline{q}$ is the single displacement for $\underline{a+e}$ or $\underline{a} + (\underline{b} + \underline{c})$

Now, apply displacements of $\underline{p}$ and $\underline{q}$ to point A. (use acetate and the arrows for $\underline{p}$ and q above.)



Do you get the same result?

Then ($\underline{a}$ + $\underline{b}$) + $\underline{c}$ = $\underline{a}$ + ($\underline{b}$ + $\underline{c}$)

This is the associative property for vector addition. Have students go through this development with other vector triples.

Relate to numbers: (5 + 7) = 6 , 5 + (7 + 6)

The commutative and associative properties together give a great deal of flexibility for vector addition.

Have students provide twelve expressions for adding three displacements $\underline{a}$, $\underline{b}$, $\underline{c}$.

($\underline{a}$ + $\underline{c}$) = $\underline{b}$, ($\underline{b}$ + $\underline{c}$) + $\underline{a}$, $\underline{c}$ + ($\underline{b}$ + $\underline{a}$), etc.

Have them draw a vector diagram for each expression and observe if the final single displacement in each case is the same.

This should give them a feeling that the commutative and associative properties yield a rather significant result.

Compare with addition of numbers having the same flexibility. Relate this flexibility to displacement of objects

D. Special Displacements, Zero and Opposite.

Think of the displacement which leaves a point in its original position.

We call this the zero displacement and use the symbol $\ominus$ to name it.

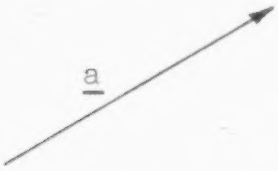
Consider the displacement $\underline{a}$ + $\ominus$ and apply it to point A. (make drawing) Do you get the same result as if you had applied only displacement $\underline{a}$? Then $\underline{a}$ + $\ominus$ = $\underline{a}$. Likewise $\ominus$ + $\underline{a}$ = $\underline{a}$.

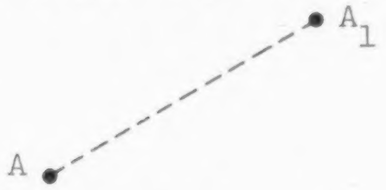
Displacement $\ominus$ is the identity displacement for vector addition.

Compare with the number 0 .

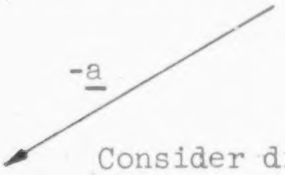
Another special displacement:

Think of a displacement $\underline{a}$ which displaces point A to A_1 as shown.





Now think of the displacement which displaces A_1 back to A. We call this displacement the opposite of displacement $\underline{a}$, and name it displacement $\underline{-a}$.



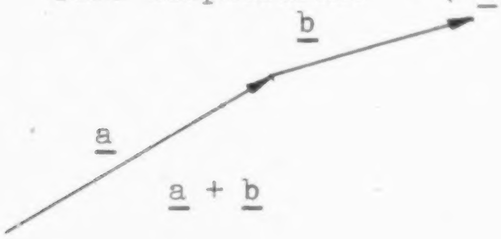
Consider displacement $\underline{a} + (\underline{-a})$ and apply it to point $\underline{A}$. (make drawing.) Does it leave the point in its original position as would zero displacement?

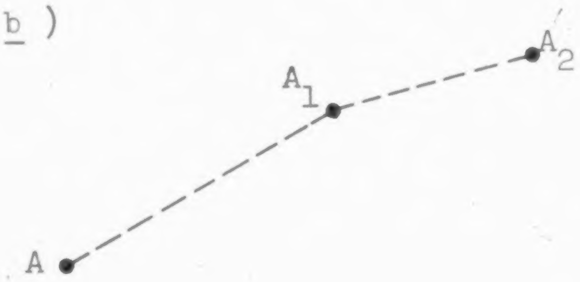
Then $\underline{a} + (\underline{-a}) = \ominus$; also $(\underline{-a}) + \underline{a} = \ominus$

Have students find and show opposites of given displacements. Also find oppoites of opposites, etc.

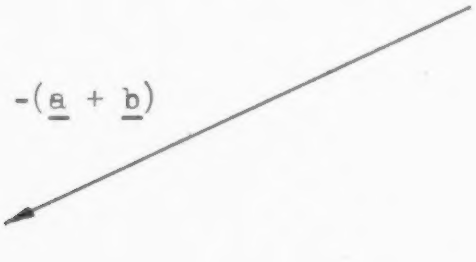
Relate to returning objects to original positions. (Reciprocating motion, pendulums, clock balance wheels)

Find displacement $- (\underline{a} + \underline{b})$



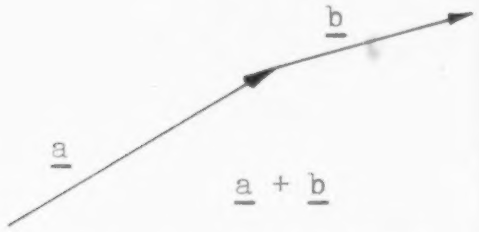


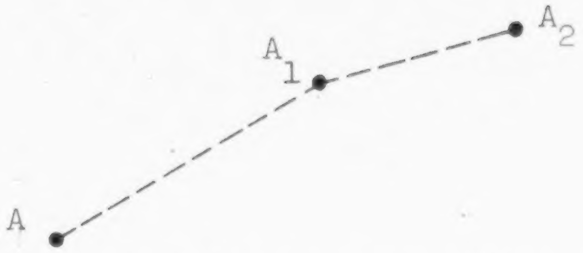
Apply displacement $\underline{a+b}$ to point A resulting in A_2. Then find the displacement that displaces A_2 to A. This is called the opposite of $\underline{a + b}$ or $- (\underline{a + b})$ and is shown below.



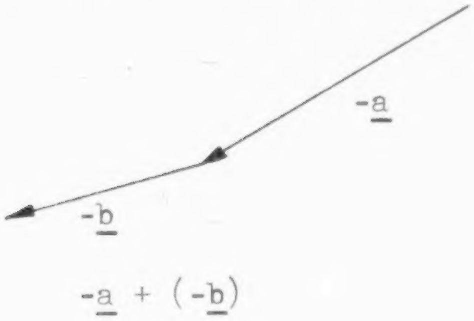
Have students do some of these problems.

Now find displacement - a + (- b)

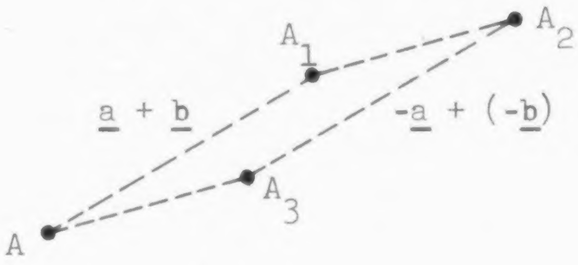




Again, apply displacement a + b to point A as shown above. The displacement -a + (-b) is represented below at left.

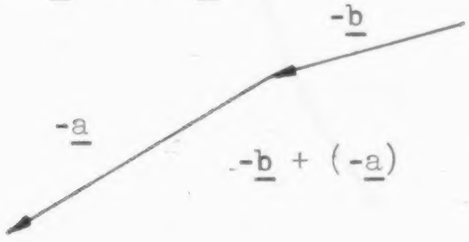


Apply it to point A_2 as shown below.



Displacement -a + (-b) displaces A_2 to A as did displacement - (a + b).

Then - (a + b) = - a + (-b)



Notice that displacement -b + (-a) when applied to point A_2 displaces A_2 to A through A_1, reversing the displacement a + b.

Relate this "back-tracking" or reversing of original displacement of objects.

E. Summary and Review of Properties of Addition.

For any displacements $\underline{a}$, $\underline{b}$, and $\underline{c}$

1. $\underline{a} + \underline{b}$ is a displacement
2. $\underline{a} + \underline{b} = \underline{b} + \underline{a}$
3. $(\underline{a} + \underline{b}) + \underline{c} = \underline{a} + (\underline{b} + \underline{c})$
4. There is a displacement $\ominus$, called the zero displacement, such that
   $\underline{a} + \ominus = \ominus + \underline{a} = \underline{a}$
5 For each displacement $\underline{a}$ there is a displacement $\underline{-a}$, called the opposite of displacement $\underline{a}$, such that
   $\underline{a} + (-\underline{a}) = (-\underline{a}) + \underline{a} = \ominus$

Relate these properties to those for the addition of numbers. Point out that these properties make the addition of displacements structurally the same as the addition of numbers (integers, rationals or reals.)

(Note: This may be the first time that students have encountered a mathematical system for something other than numbers. Make the most of it.)

F. Subtraction

(Note: The main reason for introducing subtraction at this point is to continue the structural similarity with numbers and show that subtraction is related to addition for vectors in the same way as for numbers. The physical interpretations for vector subtraction is quite limited at this level. To introduce relative displacement and relative velocity for junior high students seems questionable, Use of vector subtraction to determine change in velocity may be discussed in connection with acceleration later. In the meantime this limits applications of vector subtraction to displacements and velocities that are "in line.")

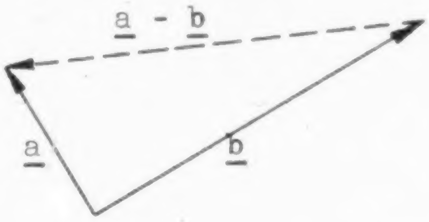
The following example show how addition and subtraction are related for numbers.

What number is 8 - 5 ? It is that number which when added to 5 equals 8. Then it is the number 3. (In formal terms, $\underline{a-b} = \underline{n}$ if and only if $\underline{n+b} = \underline{a}$.)

We define subtraction for displacements in the same way.

Displacement a - b is that displacement which when added to displacement b has the same result as displacement a.

This can be shown by the following vector diagram.

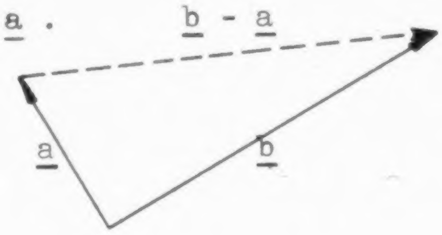


Given: displacements a and b as show.

Find: displacement a - b

The dotted arrow represents displacement a-b because it shows that the addition of a-b to b results in a .

Now find b - a .



The dotted arrow represents displacement b - a because it shows that the addition of b - a to a results in b.

Have students do this type of exercise with given pairs of displacements. Emphasize connection with addition. Also have students find displacement a + b along with displacement a - b. Note that subtraction is not commutative.

The physical counterpart of this would be as follows: two objects start from some point; object A is given displacement a; object B is given displacement b. a - b is the relative displacement of object A ith respect to object B in the process. b - a is the relative displacement of A with respect to A.

This would probably be a difficult idea to get across to junior high youngsters, even with "in line" displacements. There is some work on this later with velocities.

It might be possible to find applications from economics and linear programming that would be helpful here..

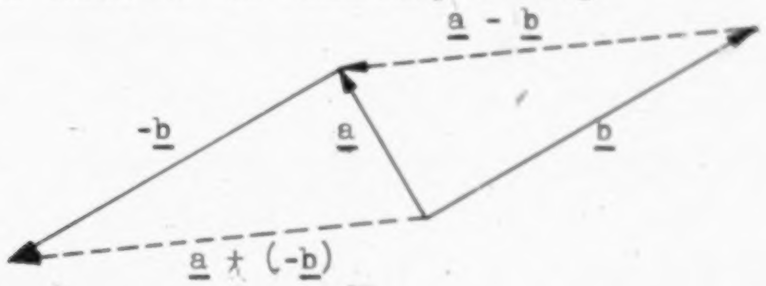
Problem: Spaceship docking is accomplished by ground telling spaceship where it is with respect to the trailer. It is hard to find things in space just by looking. Ground launches trailer and keeps track; also it knows the location of the spaceship. It tells the spaceship where to look for the trailer.

Showing subtraction another way:

Since each displacement has an opposite, subtraction can be shown another way.

Have students show displacement a - b (see drawing below) in the way they have just learned. Then have them represent displacement

$\underline{a} + (-\underline{b})$ as shown in the following drawing.



$\underline{a} - \underline{b}$
$-\underline{b}$
$\underline{a}$
$\underline{b}$
$\underline{a} + (-\underline{b})$

Do the displacements $\underline{a} - \underline{b}$ and $\underline{a} + (-\underline{b})$ have the same result; that is, the same direction and magnitude ?

Then $\underline{a} - \underline{b} = \underline{a} + (-\underline{b})$.

Give exercises asking students to show subtraction in these two ways.

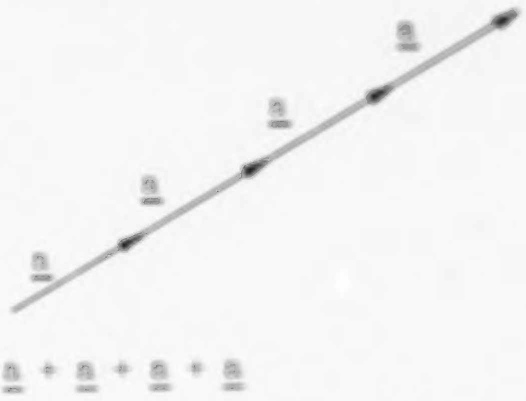
Show analogy to numbers $5 - 8 = 5 + (-8)$.

Show that $\underline{a} - \underline{a} = -$

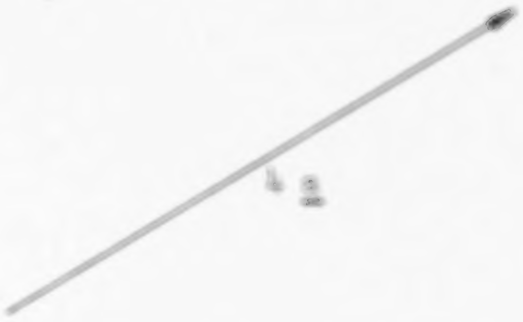
IV. Multiplication of Vectors by Scalars

A. Developing meaning

Multiplication by scalars can be introduced through repeated addition.



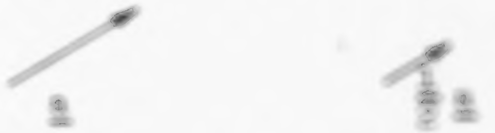
Have students represent the repeated addition of a displacement such as a+a+a+a



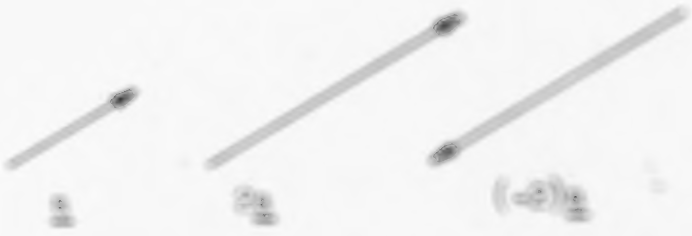
After having them do a few, including some that require extensive repeated additions that may run off the paper, suggest that these laborious repeated additions can be replaced by a single displacement having the same result.

Note displacement 4_a above which describes the same displacement as a + a + a + a . Note that 4 is a scalar and a is the vector, and that they are combined in this special way which is called multiplication of a vector by a scalar.

Give exercises having students represent, for example, b + b + b + b + b + b and 6b. Have them describe but not draw displacement [illegible]b.



Develop meaning for $\frac{1}{2}$ a, as shown left, and for $\frac{1}{4}$ a, .7 a, 2 $\frac{1}{2}$ a, 1.8 a, etc.
Include 0 a and 1 a.



Develop meaning for multiplication by negative scalar as show for (-2) a.

Note the following distinction:

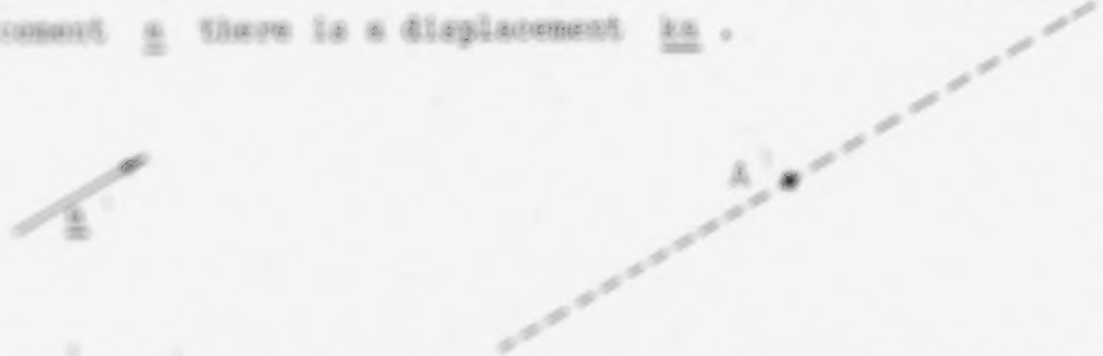
$(-2)\,\underline{a}$: multiplication of $\underline{a}$ by negative 2

$-(2\,\underline{a})$: the opposite of displacemtn $2\,\underline{a}$

These displacements have the same result, $(-2)\,\underline{a} = -(2\underline{a})$. Then we can think of displacement $-2\underline{a}$ either way.

Have students make representations of several displacements of this type, including $-\frac{7}{4}\,\underline{a}$, $-3.2\,\underline{a}$, etc.

Develop idea through the activity below that for any scalar $\underline{k}$ and displacement $\underline{a}$ there is a displacement $\underline{ka}$.



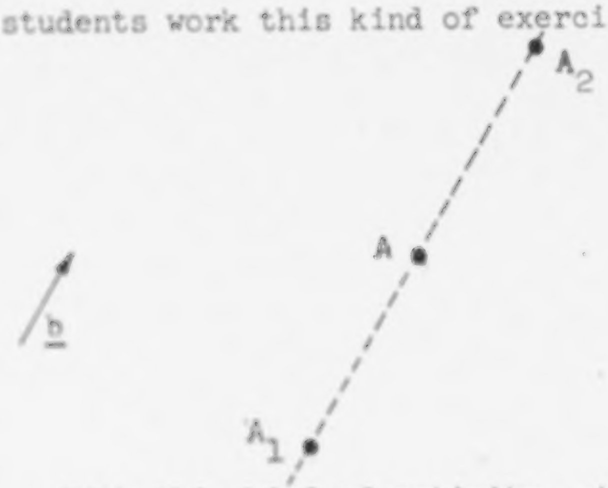
Given displacement $\underline{a}$ and point A. Have students mark points showing various displacements of A such as $3\,\underline{a}$, $-1\frac{1}{2}\,\underline{a}$, $\frac{9}{4}\,\underline{a}$, $-\frac{5}{3}\,\underline{a}$, etc. [illegible] and that for every scalar k there is a point on this line for displacement $k\underline{a}$ applied to point A.

Now turn this around. Choose an arbitrary point on this line for a displacement of A. Is there a scalar k such that $k\,\underline{a}$ is the displacement applied to A ?

We assert that there is, although in finding k, we often have to resort to an approximation.

This develops the idea of a one - to - correspondence between the points on this line and scalars.

Have students work this kind of exercise:

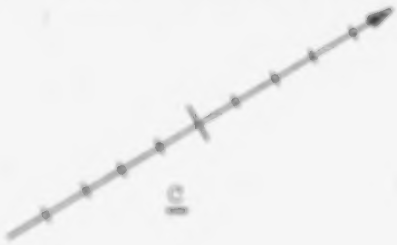


Given displacement $\underline{b}$ and point A. Find displacement $k\underline{b}$ which displaces A to A_1. Also find displacement $k\underline{b}$ which displaces A to A_2.

Students find k approximately.

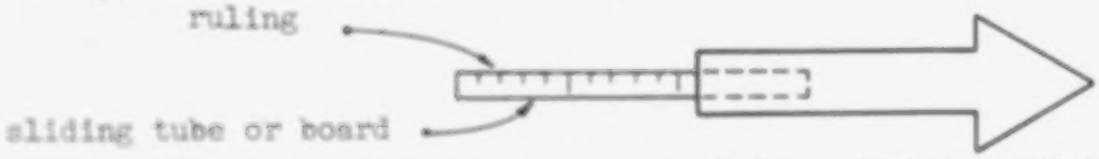
With this kind of activity, students will soon realize that this is similar to measurement, using $\underline{b}$ as a unit.

To make these activities easier for students, it might be well to consider scaling the vector representations of the displacements so that they will have a scale available.



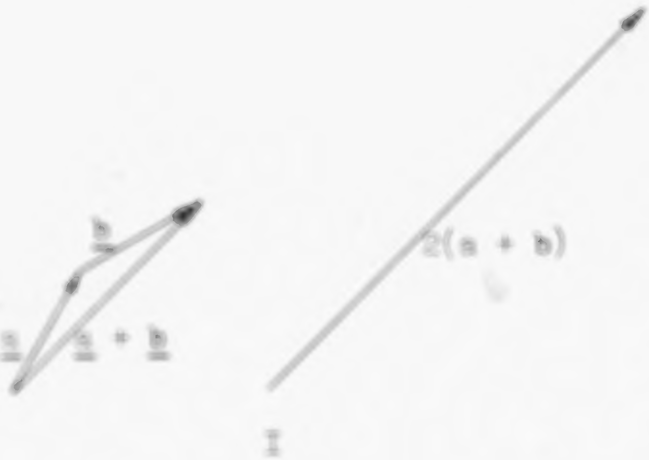
At left shows the vector representation of displacement $\underline{c}$ scaled in tenths. Also the use of the rulings on ruled paper might be considered.

(It might be worthwhile to develop a telescoping vector.

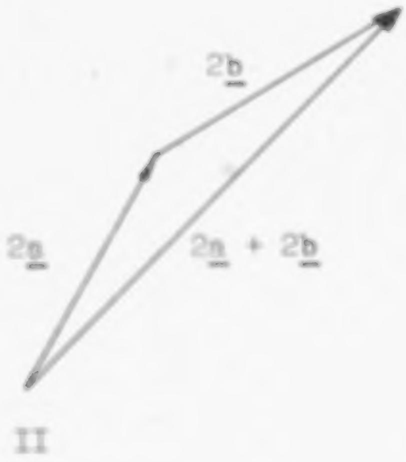


Felt could let it stick to paper or acetate. A blackboard vector would be handy.)

B. Some Properties of Multiplication By Scalars.



I



II

The first drawing shown on bottom of preceding page, shows the construction of displacement $2 \; (a + b)$. The second shows the construction of $2 a + 2 b$. Comparison of these two representations shows that they have the same direction and magnitude.

then, $2 (\underline{a} + \underline{b}) = 2 \underline{a} + 2 \underline{b}$.

Have students make similar constructions. Try $3 (\underline{a} + \underline{b})$ and $3 a + 3 b$, $\frac{1}{2} (a = b)$ and $\frac{1}{2} a + \frac{1}{2} b$.

Lead to general pattern: $k \; (\underline{a} + \underline{b}) = k \underline{a} + k \underline{b}$. Note that that + sign is for vector addition.

Next have students make a representation of displacements $(2 + 3) \underline{a}$ and of $2 \underline{a} + 3 \underline{a}$, and compare. Does $(2 + 3) \underline{a}$ equal $2 \underline{a} + 3 \underline{a}$?

More investigations of this type should lead to the genera result: for scalars k and m . and displacement $\underline{a}$

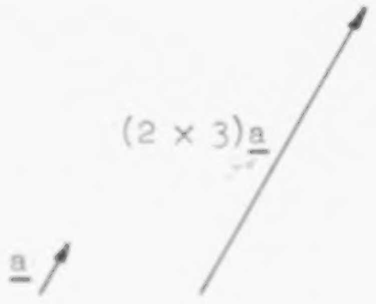
$$(k + m) \underline{a} = k \underline{a} + m \underline{a}$$

Note the use of + sign for addition of scalars and for addition of vectors.

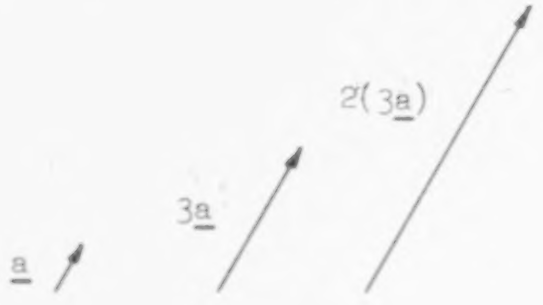
Have students study these two expressions to determine what they mean:

$(2 \times 3) \underline{a}$ $\qquad\qquad$ $2 (3 \underline{a})$

Then have them construct representations of these displacements in stages, and compare results.



I



II

Does $(2 \times 3) \underline{a}$ equal $2 (3 \underline{a})$?

Note the multiplication of two scalars (k X m) and the multiplication of a vector by a scalar $\{ (k \times m) \underline{a} \}$

Summarize these properties by discussing different ways of constructing and naming a displacement.

For example, have students describe the construction of :

$12\,\underline{a}$

$(4 \times 3)\,\underline{a}$

$4 (3\,\underline{a})$

$(4 + 8)\,\underline{a}$

$4\,\underline{a} + 8\,\underline{a}$

etc.

also

$5 (a + b)$

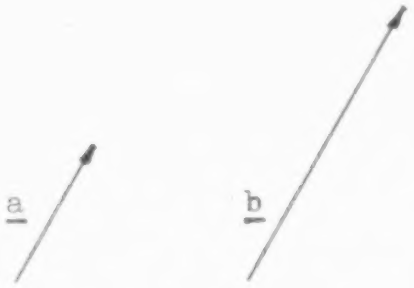
$5\,\underline{a} + 5\,\underline{b}$

V. Developing a System for Describing Displacements

A. Describing a Displacement in Terms of Two Given Displacements.

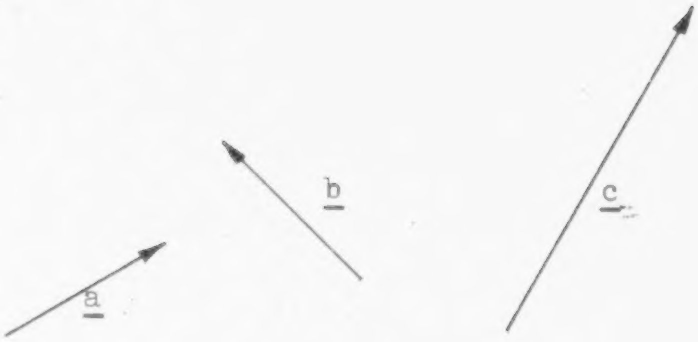
After seeing that a displacement may be described in different ways, the students should be ready to discuss the important idea of describing a displacement in terms of other displacements.

(Note that the ideas discussed here are those of linear dependence and independence, and a vector basis, but are not labeled as such.)

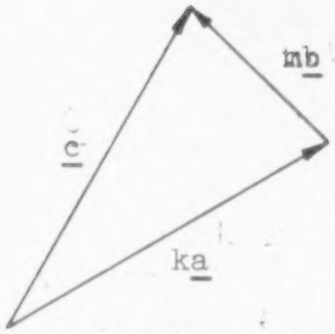


Shown at left is a case in which one displacement, $\underline{b}$, can be expressed in terms of another displacement, $\underline{a}$, as follows: $\underline{b} = 2\,\underline{a}$

Whenever this occurs for two displacements $\underline{a}$ and $\underline{b}$ such that $\underline{b} = k\,\underline{a}$, they are said to be parallel.

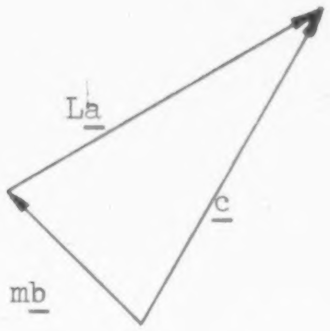


Suppose $\underline{a}$ and $\underline{b}$ are two non-zero non-parallel displacements (one cannot be expressed in terms of the other), and $\underline{c}$ is another displacement. Express $\underline{c}$ in terms of $\underline{a}$ and $\underline{b}$.



c = 2a + 1$\underline{b}$

Draw displacement $\underline{c}$. Then sketch lightly a line for $\underline{a}$ and one for $\underline{b}$ to find where they meet. Draw k $\underline{a}$ and m $\underline{b}$ as shown. Then find what the scalars k and m are.



$\underline{c}$ = 1$\underline{b}$ + 2$\underline{a}$

This shows another way of drawing k $\underline{a}$ and m $\underline{b}$ and finding the same result for scalars k and m.

Present other displacements to be expressed in terms of displacements $\underline{a}$ and $\underline{b}$. Help the students get started on the constructions.

Include a case in which the scalars k and m are approximated. Also include cases where k or m is negative.

Provide enough experience so that students see that there is a general result:

For any displacement, say $\underline{c}$, there are sclars k and m such that

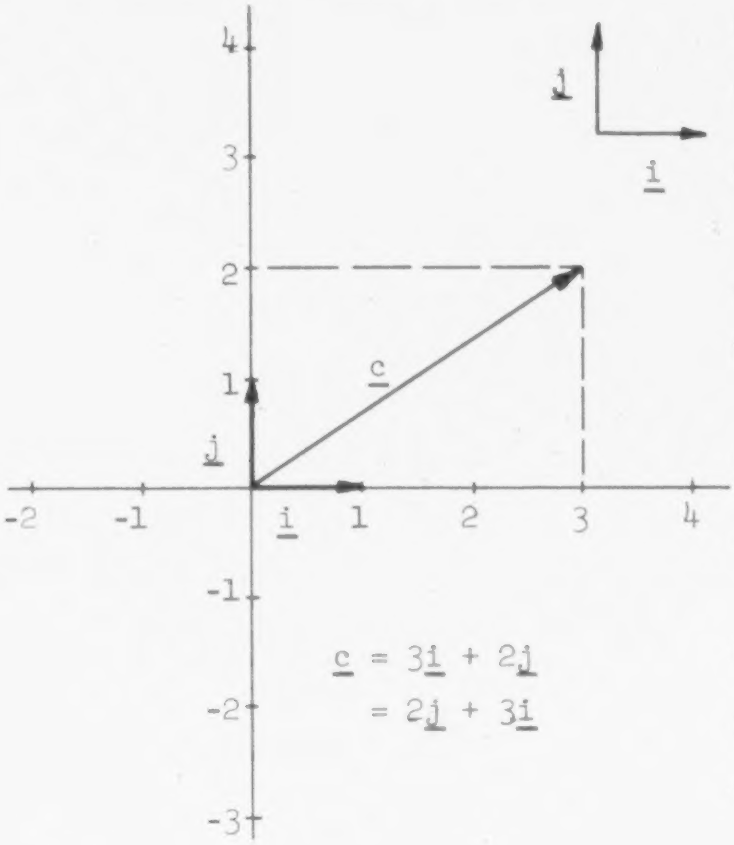
$$\underline{c} = k\,\underline{a} + m\,\underline{b}$$

Start with another pair of non-zero, non-parallel displacements and lead to the same general result: any other displacement can be expressed in terms of these two displacements.

B. A Convenient Way of Naming Displacements

The idea that any displacement (in 2 dimensions) can be expressed in terms of two non-zero, non-parallel displacements, can be used to set up a convenient way of disignating displacements.

Ask students to provide suggestions. Keep raising question of convenience. Give hints when needed.



$\underline{j}$
$\underline{i}$
4
3
2
1
-1
-2
-3
-2
-1
1
2
3
4
$\underline{c}$
$\underline{j}$
$\underline{i}$
$\underline{c} = 3\underline{i} + 2\underline{j}$
$= 2\underline{j} + 3\underline{i}$

Show two orthogonal unit displacements. Why at a right angle? Why both unit displacements? What are the advantages?

A unit suggests a scale. Develop one. Why can we go in negative direction also?

Students begin to see the familiar coordinate system.

Now, discuss expressing another displacement, say $\underline{c}$ as shown, in term of the two unit displacements, $\underline{i}$ and $\underline{j}$.

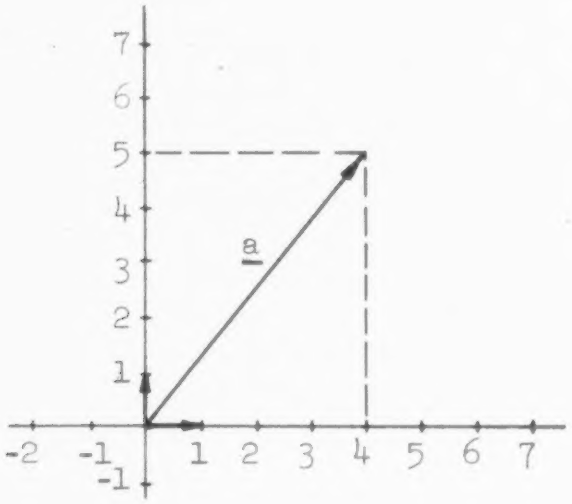
Have students develop the two expressions shown, and show how they are obtained: $c = 3\,\underline{i} + 2\,\underline{j}$

$= 2\,\underline{j} + 3\,\underline{i}$

Then provide other displacements to be expressed similarly.

Next, provide a variety of expressions, such as $\underline{a} = 5\,\underline{i} + (\,-4\,)\,\underline{j}$ and have students provide the drawings related to each.

Keep the scalars rather **simple at first** (possibly limited to integers), then bring in rational to show generality of this way of describing vectors.



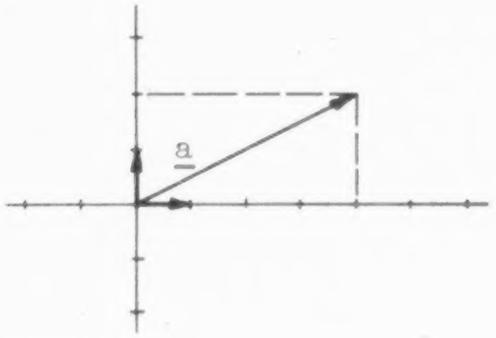
Given displacement $\underline{a}$

$$\underline{a} = 4\ \underline{i} + 5\ \underline{j}$$

Note that the displacement could be described by just naming the scalars in an agreed upon order.

a : (4,5)

Give students experience in using this notation.



Given displacement $\underline{a}$, (4,2). Apply this displacement to point A.

A •

Optional:

1. Use this new notation and go back to consider addition of vectors, and multiplication by scalars.
2. Position vectors.
3. Dot product.

Part II -- Enlarging the Physical Development

I. Scalars

A. Have students look at physical quantities which are completely specified by a "number" and a unit (volume, mass, time, temperature, period, etc.). The discussion of physical scalars should not be general or abstract. The point to emphasize is that there is more to a volume than is indicated by considering porous materials like sponges or coke. In this case there are two volumes of possible interest: the air volume and the solid volume. In each sclar considered the student should see and hadle examples of the material being discussed. It would be helpful if the discussion included how one would assign a value to the quantity being discussed. It would be very desirable to discuss the whole idea of measurement.

B. The physical quantities selected above are used to introduce the algebra of scalars. Since the algebra of units is a very confusing area the whole problem may be sidestepped by introducing the mathematical model. If each set of scalars is always reduced to the same unit basis then the mathematical model becomes the "number" associated with the physical scalar. The mathematical model is then a set of numbers. The idea of forming a mathematical representation of a physical quantity or system helps a great deal later in the development of vectors. Multiplication of a vector by the scalar produces another physical quantity. The mathematical model will have different meaning. As an example, Newton's second law ($F = ma$). Force and acceleration are physically different but the mathematical model in some cases will be the same. Introducing the mathematical model at this time should pave the way for its use at these later times.

If some physical quantities are listed which are not scalars it can be established or made reasonable that a description of the " real world" may need different or entended mathematical ideas.

Displacement in the form of (chess or checker moves) can be introduced as an example and can be tied in with the following vector development. There are probably better examples than the board moves in checkers but they should be something the student can actually *do*. The examples should *not* be thought experiments s.e. hypothetical trips etc.

II. Vectors

There is a class of physical quantities which need for their description a number and a direction (also units). This section will define a vector and develop the algebra of vectors. Since displacement is an idea which is easily grasped the difinition will be formulated in these terms. The formulation of a mathematical model from experience in the "real world" should be carried throughout.

A. The definition of a vector follows from "physical" experience. The use of a frosted acetate overlay allows the student to develop for himself the basic properties he associates with vectors. The formal definition and representation $\longrightarrow$ as a directed movement of a plane should be quite logical and real. This may be a good place to have the student tentatively identify other physical quantities with vectors -- velocity, acceleration, force. The question is raised how do we determine whether or not any of these quantities can be represented by vectors. To do this we must establish the algebraic properties of vectors.

B. Return to the acetate and establish the rules for vector operations of addition, multiplication by a scalar (number), etc. Introduce additive inverse, zero element, maybe unit vector, Follow these with the CAD laws. All of this should be done with acetate by the student -- operations are to be made reasonable.

There should be no formal mathematical proofs. It should be emphasized that the displaced acetate is the physical world and *can* be represented by mathematical systems of vectors.

C. Using experiments have the student discover and test other physical quantities as to the feasibility of using a vector model to represent them. In this section there should be come examples which do not lend themselves to vector representation. Pressure might be a reasonable example, electric current another. As an example of vectors, velocity seems to be quite straightforward. Acceleration whould be considered. Concurrent forces could also be introduced. If forces and acceleration can be introduced than scalar (physical) multiplication can be considered in an operational sense, s.e. mass multiplying acceleration is equal to force.

D. Since multiplication by a scalar is introduced in the section on displacement the extension of this in physics could be very helpful. In displacement the multiplication by scalars is essentially a change in scale. No new physical quantity is generated by this multiplication. There are some cases in which multiplication by a scalar generates a new physical quantity. Momentum, for example, is formed by the scalar multiplication of mass and velocity. Force is related to acceleration multiplied by mass. This approach could lead a long way to establishing the operational ideas in science. Cause (force) and effect (acceleration) can be established at this time. It is here that the idea of a mathematical model helps considerably. The force diagram and acceleration diagram are different in the real world but to the mathematical model they are equivalent. *Note:* If Newton's second law is the resultant force which is the cause of the acceleration, physically it is hard to find the resultant force.

Representing the forces by a vector model we can quite easily find the sum or resultant vector. This vector referred to the physical situation can give the magnitude and direction of the acceleration.. Another utilitarian aspect of the mathematical model comes when we ignore rotation. The model does not have to correspond directly to the physical case. If rotation is to be ignored, the mathematical model will have all vectors meeting at a point. This model will describe the motion of the bodies center of mass. Sometimes this is enough for the physical situation and makes life much simpler. Forces should be considered only after the student is fairly familiar with vectors.

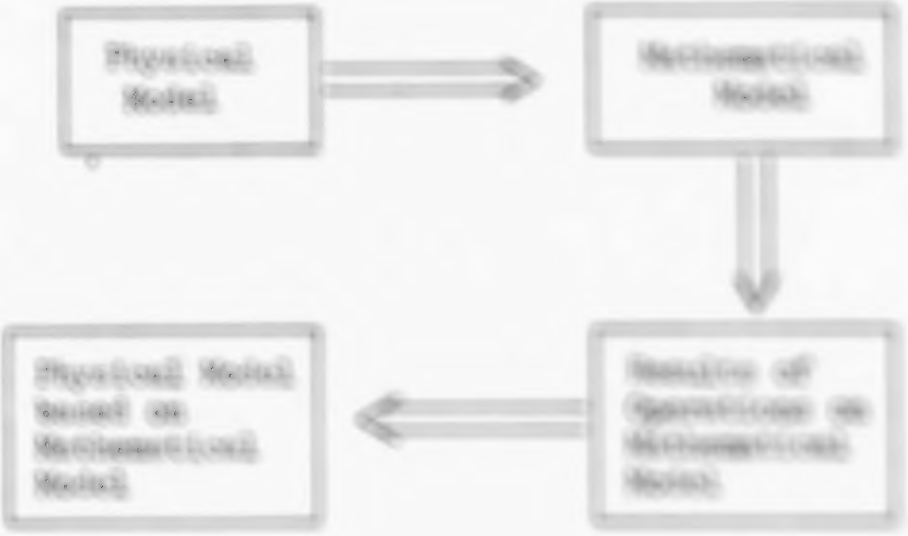
E. The possibility of introducing dot products should be investigated. This is not easy. The mathematical concepts are hard and experiments involving energy are difficult to make work. They are possible, however.

Also the feasilility of rotations must be considered. This means not only rigid body sort of rotations but also point rotations. PSSC does not discuss rigid body rotations or torques.

F. Added note to outline.

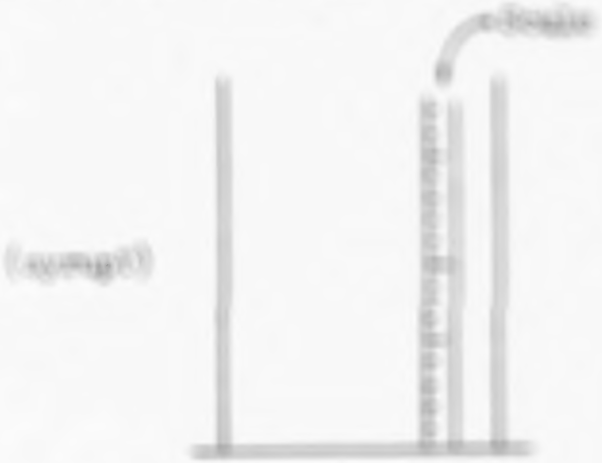
Introduce the idea of constructing a mathematical model from kinematics (displacement, velocity, acceleration). Kinematics does not clutter up the situation with physical constructs like force, momentum, and energy. This allows a student to study physical systems without considering the cause of the systems behavior. Wher algebra of vectors and development of models is fairly well established then introduce physical constructs (force etc.) and cause and effect

A possible "flow" diagram might aid in establishing the procedure.


Physical Model
Mathematical Model
Results of Operations on Mathematical Model
Physical Model based on Mathematical Model

III. Velocity as a Vector

Investigate the falling sphere case:


Scale
(syrup)

Sphere 1 ○ fast

Sphere 2 ◎ slow

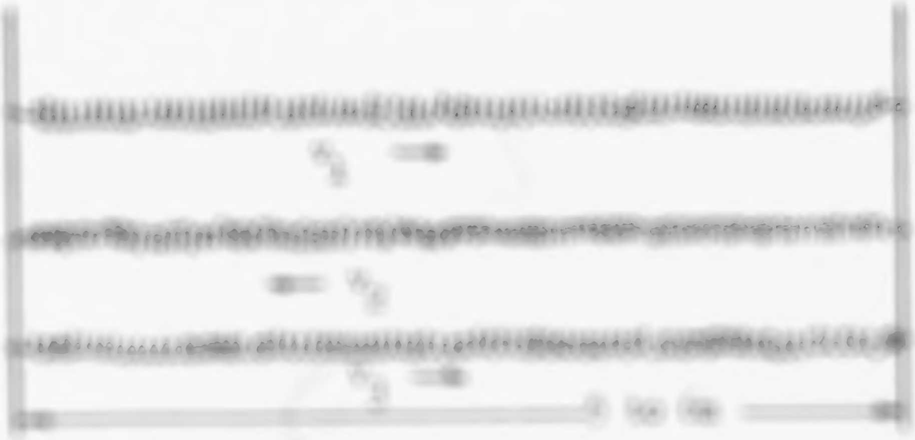
Drop each sphere by itself and observe its motion. The student [illegible] said that each sphere falls at a constant rate or velocity. There are a number of methods by which one can describe the motion of the ball. A set of ordered pairs can be obtained by observing the distance covered by the sphere from some origin during various time intervals. A graph of these ordered pairs can be used to describe the motion of the sphere. If, instead of the sphere falling, a bubble is introduced at the bottom and allowed to rise we have a different situation. Again measurements can be made and a graphical model developed to describe the motion. Another possible way which conveys as much information is to state the rate at which the ball moves. Since the object may move either up or down a direction should be assigned to the rate. This discussion leads one to suggest a vector representation may be possible.

ERIC

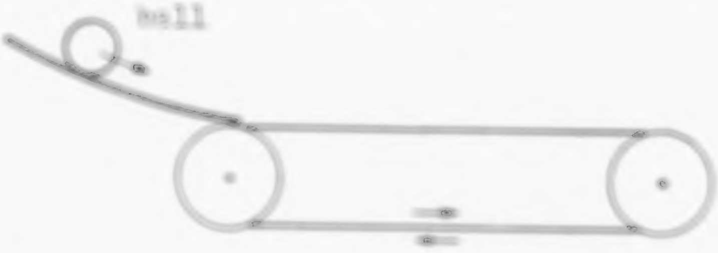
At any position along the tube the sphere falls at a constant rate (speed) and is moving either up or down. Therefore let us make a mathematical representation of this motion by using a vector of a length proportional to the rate and pointing in the direction of the motion. This does not establish that a vector representation is possible for velocity but we can test it. Do velocities obey the algebra of vectors? Let us describe the motion of two balls in the column of liquid. Let ○ represent a fast ball and ⦶ represent a ball which moves at a slower rate than ○. Release the ⦶ and time its fall over a known distance on the scale. From this data the velocity of the ball corresponds to a vector with a suitable magnitude and pointing downward. Now perform the experiment with two balls. Release the ⦶ first and let it drift downward to a prescribed mark on the scale. As it moves past the mark release the ○ and start the timer. When the two balls pass one another the fast ball has traveled a distance relative to slow ball equal to their initial separation. Dividing this distance by the time gives the relative velocity of the two balls. This relative velocity can hopefully also be represented by a vector. Our two experiments have provided the velocity of one object referred to the scale and velocity of a second object with respect to the first object. Our vector representation of these velocities says that the addition of the ⦶ velocity and the relative velocity should give the velocity of ○ referred to the scale. Perform the sum of the two velocities and check by a third experiment: Dropping ○ by itself and computing its velocity.

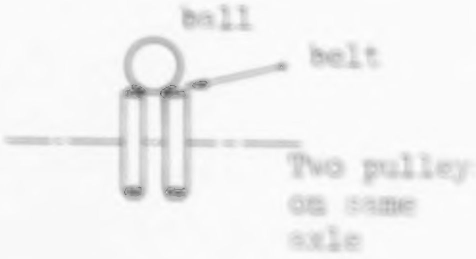
The introduction of subtraction can be accomplished by inserting a straw in the dropping tube and blowing a bubble. An alternative way to produce a rising sphere would be to seat a light ball on the end or the straw and push it to the bottom. The rising sphere is considered to have a negative velocity. Measurement procedures similar to those described in the previous section yield the necessary velocities. By using three balls it can be demonstrated that the CAD laws hold. The motion of a sphere in in a viscous medium is used only as one of any possible means of introducing students to vector ideas. Other possible schemes that seem possible are:

1. Longitudinal pulses propagating down springs. A very weak spring under low tension will have a reasonable velocity of propagation.



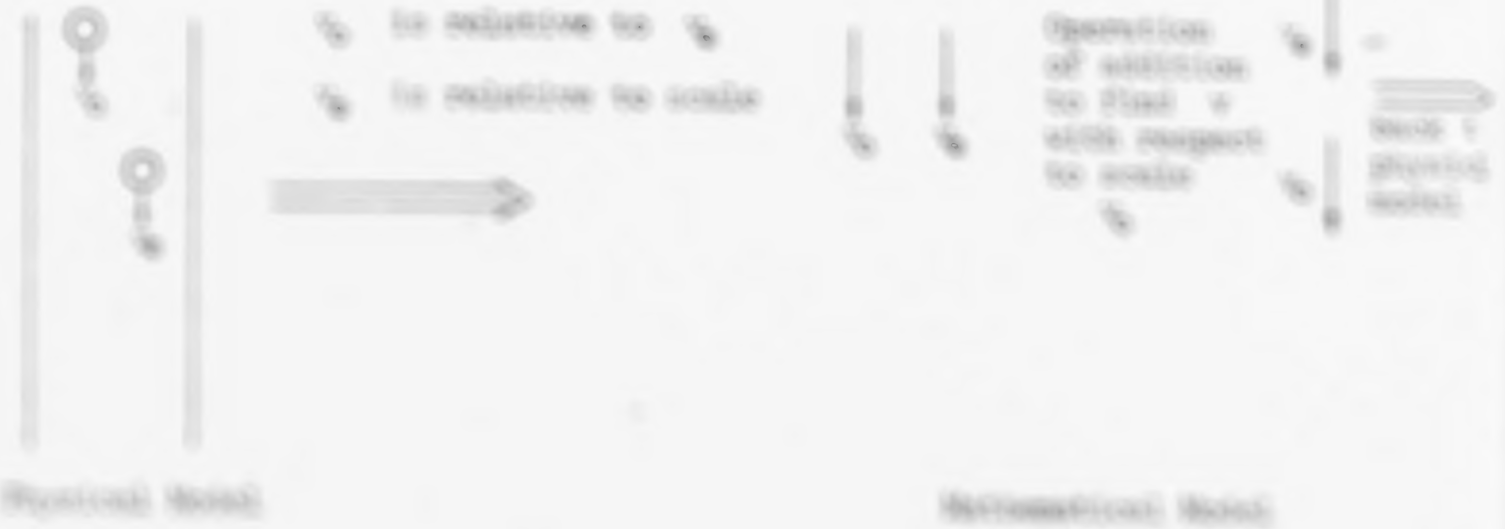
2. A moving endless belt with a ball rolling on it.



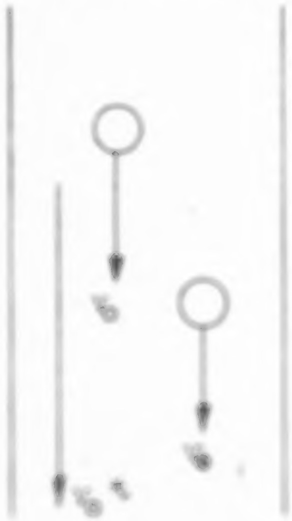

ball
belt
Two pulley on same axle

Ramp can be formed by bending a plastic ruler. It is attached to the system and allows smooth injection of the ball onto the belt. The ramp should permit various velocities to be reproduced. Tapes cemented to the flor and belt make distance measurements easy and from corresponding time measurements the velocities can be calculated.

When the student has convinced himself that velocities can be represented by vectors he should proceed to develop the mathematical model of the physical situation he has been observing.

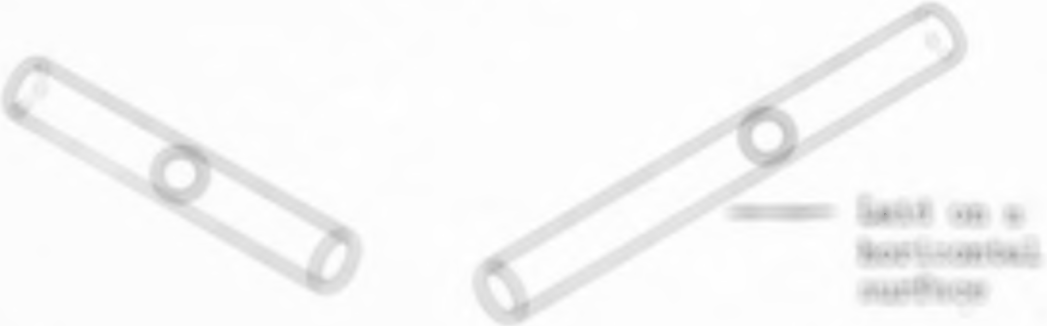

is relative to
is relative to scale
Operation of addition to find v with respect to scale
Back to physical model
Physical Model
Mathematical Model

The result of the vector addition is referred to the physical situation.



For the case of the rising ball the operation is of subtractions and follows the above sequence of operations. The mathematical model is not tied to any reference plane since there has been no discussion of bound vectors. The operation (s) to be applied to the mathematical model are inferred from the physical situation. It is in the physical world that we associate velocities with scales and reference frames. It is also possible in the mathematical model to have reference frames, but at this stage a reference to axes would probably just confuse the issue. It should be mentioned also that we are working in velocity space. Distances should not appear on the physical model.

The previous discussion has been concerned with in line velocities. It may be worth while to extend the treatment of velocities to oblique motions. A simple method may be through the use of clear lucite tubes on aluminum vee rails with marbles. These systems should be fairly friction free and the velocities constant.



Even if friction is a problem a small piece of tape at one end of the tube or rail should establish a uniform velocity. The analysis should follow that discussed above or in the section on Displacement.

IV. Acceleration

If acceleration is introduced through in line motion some conceptual difficulties can be avoided. In more general motions acceleration occurs both from a change in length of the velocity vector and also from a change in direction. There is much to be said for following the pattern used in physics of discussing accelerations where analytic length of the velocity vector changes and then following with changes only in direction. Consideration of motion where both changes occur is left to the last. The standar ballistic problem of shooting a projectile in a horizontal direction also presents unnecessary complications. It combines both a constant horizontal motion with a vertical linear acceleration. The complication in this situation comes from resolving the vectors into orthogonal components -- vertical and horizontal. At this stage the vectors should be left free.

The question arises as to how far to carry the investigation of acceleration. A change in length of the velocity vector will establish in the students' minds that acceleration can be represented by a vector. In a physical sense the change in direction is important but at the early stages not completely necessary. If straight line motion becomes the common thread throughout the book, changes in direction should not be attempted.

There exists a great problem in measuring acceleration. At present we do not have simple acceleration measuring instruments. The standard and maybe the best treatment probably lies along using a tape which is marked by a spark at equal time intervals. Successive differences in distance and the time interval gives the acceleration. An Atwood's machine or inclined plane will give low accelerations and make measurement quite easy. Also they operate with constant acceleration and hence average accelerations do not enter.

Hopefully there are more imaginative ways of introducing accelerations. The major criterion for an experiment should be that the measurements should be simple and obvious and the accelerations easily calculated. The vector characteristic is the important item, not the experiment or the physics. Finally, the analysis can be carried out in a manner similar to displacement and velocity.

V. Forces

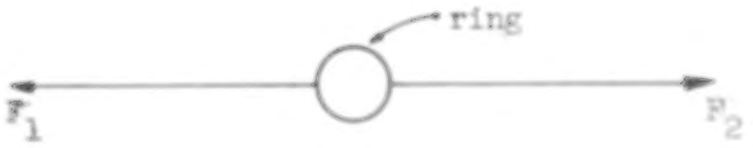
In this experiment the student is limited to a discussion of concurrent forces. Omit for the present the observation of forces exerted on extended bodies -- distributed forces. A force table is an easy piece of equipment to work with and involves only concurrent forces. In describing the operation of a force table the student should understand what is meant by strings. A string can transmit a force only along its length; s.e. it is incapable of sustaining any shear forces.

1. Forces in general.

Forces are associated with dynamics and for the first time we are considering physical quantities which cause something to happen. This is different from merely describing the motion or behavior of a system. In describing physical phenomena we introduce new quantities which are considered to be the cause of something happening. Force is such a quantity. If a system is at rest or in uniform motion and this state is observed to change a net force is acting on the system. Later we can consider quantitatively the effect of one force on the motion of a body.

Forces are subtle physical quantities and the development of mathematical models corresponding to general situations is quite difficult. By this we mean a generalization or inference from a specific example is liable to lead to quite erroneous conclusions.

II. Equilibrium on a force table.



If we consider the case where the two strings exert equal and opposite forces on the ring there is no motion. Before the strings were attached there was no motion and no net force on the ring. There still is no motion and the state of the system remains the same, hence there is no net force acting on the system. The problem is now to describe the physical system by a mathematical model. If we remove F_2 the ring slides to the left. Removing F_1 allows the ring to move to the right. If we change either F_1 or F_2 one at a time similar changes in the state of the ring occur. Since both direction and magnitude are involved we can try as a possibility a vector model. Let F_1 and F_2 correspond to two vectors which add to give the null vector. If vector F_1 is shortened, then the sum does not result in the null vector and we have a new vector to the right which gives the correct direction to the motion in the physical case. Shortening F_2 also yields a correct result. If one of the forces is replaced by two vectors, the various algebraic rules can be established.

This analysis does not show that forces can always be represented by a vector model of this type. Forces cannot in general be moved arbitrarily. The physical situation determines where forces are applied and the points of application must be considered in discussion of the detailed motion. The force table has no problems along these lines since the forces always act at a point. Finally, the analysis should be extended to forces acting at various angles.

RESTRICTING AND FREEING THE INTUITION

H. O. Pollak

A comment in the "Suggestions to SMSG on the Secondary School Mathematics Program" is that algebraic topology is the applied mathematics of the future. I am not sure of the immediacy of this threat; at the present I know of the applications to physics, like the many-body problem and general relativity, the many diverse applications of graph theory, uses of fixed point theorems in engineering and economics, applications of cohomology theory to the analysis of structures, as well as attempts to use topology in understanding large electrical networks. However, the question of building up some intuition on topological notions somewhere in the secondary school is, it seems to me, wide open, regardless of the imminence of a flood of applications. There are a lot of interesting concepts: dimension, deformation, orientation, fixed points, triangulation, homotopy, cutting and pasting, connectedness and multiple connectedness, open, closed, compact, curves--open and closed and arcs--surfaces, knots, geodesic, and so forth. How important is it that everyone have a feeling for some of these things?

Just in case we don't feel like solving this problem, let's generalize it. We had quite a discussion about automatically restricting a student's intuition by always using rectangular coordinate systems. How much more do we restrict the intuition by always using the invariants of Euclidean geometry! Anyway, how do you tell when an implied restriction of the intuition is a good thing and when it is a bad one? The first nonrational numbers the student sees are $\sqrt{2}$ and π, but we are careful to keep open from the very beginning that there are more. From the beginning of SMSG 7 we draw a positive number line with an extra stub to the left. We make sure that our first examples of functions as functions are not all given by simple formulas or are even all continuous. But when we introduce complex numbers, I don't recall that we do anything at all to keep the intuition open for quaternions. The beginnings of plane trigonometry ignore spherical, and I am not sure that there are any discovery exercises for nonEuclidean geometry in SMSG 11. Each of these choices seem immediately intuitively correct to me. Why? How do I tell a restriction on the intuition I like from one I don't like? Future importance of the generalization, ease of keeping the possibilities open, applicability of the more general concept all have something to do with it but don't seem to be the whole story.

The question of the long uninterrupted stretch of rectangular coordinates, incidentally, raises the question of local coordinates. In fact, every application of coordinates I can think of has the feature that the coordinate system is only locally valid. "Two aisles up and three aisles over" in the supermarket is not supposed to take you through the wall into the laundromat next door. The numbering system on El Camino Real begins over again in each community, to my original confusion. In Michigan there is a global numbering system on major roads which is interrupted by a local numbering system in incorporated units and resumed when you get out into the farms again. When we count tree rings, the very deformation of the polar system gives us significant information. I think maybe a case can be made for some intuition-building for local coordinate systems. When is it legal to think of the earth as flat? When is noon at the North Pole?

In SMSG [illegible] last year, my son had a certain amount of trouble with one particular aspect of graphing: How do you pick the origin and the scale on the little piece of graph paper you get for doing the homework so that the problems all fit and don't run over into other problems and are not so small that you can't see what you are doing? With the usual introspection, I find that in my own work I always make a quick judgment on this point before I start plotting. We never teach the kids, as far as I can remember, how to make such a judgment. This is both another example of the local coordinate question and also an application of approximations.

ON THE INTRODUCTION OF MATHEMATICAL CONCEPTS

Hassler Whitney

We shall illustrate some of the ways in which a question may be posed or a system may be set up for study, which will lead in a natural manner to the elucidation of concepts. The important thing is first to understand the workings of a simple situation, then to pull out some underlying notions. The notions thus will be understood before being formulated.

Just as in mathematical research, some questions may lead to a variety of concepts and topics. Each of these is often worthy of study. Then these topics may be compared and their relations studied. Thus a much fuller understanding of the general situation becomes possible.

Over a month, term, or year, one wishes to cover certain topics. If one is led to study and comprehend various notions and their relations without at first demanding that they occur in a certain order, it is then easy to summarize the principal facts and thus present the final material in an orderly fashion which will be understood rather than memorized.

I. Some Topics from Number Theory

1. Divisibility. Let us look at a multiplication table and ask some natural questions. For instance, what numbers appear in this table? All (natural) numbers, clearly. But this is because of the first row and column. Suppose we discard these. Then we may find numbers not appearing; for instance

1	2	3	4	5	6	7	8
2	4	6	8	10	12	14	16
3	6	9	12	15	18	21	24
4	8	12	16	20	24	28	32

$$2, 3, 5, 7, 11, 13, \ldots$$

Some numbers appear several times in the table. When does this occur? They are answers to different multiplication problems: $12 = 2 \cdot 6 = 3 \cdot 4 = 4 \cdot 3$, etc.

This suggests a definition: The number n is composite if there exists numbers a and b, both smaller than n, such that n = ab.

Theorem 1. 7 is not composite, for trying 2, 3, 4, 5, 6 none of these works.

Theorem 2. 9 is composite, for 3 is less than 9, 3 is less than 9, and 3 · 3 = 9.

Remark: "There are two numbers a, b" needs comment. The commonly used "two" is not meant in the strict sense.

Remark: After a few such examples, one may point out the use of "there exists" and "for all." In proving 9, we find 3; in proving [illegible], we find 2, 3, 4, 5 in proving 7 we find nothing. In the latter case all such numbers fail. Negations of quantifiers are already introduced in a [illegible] manner.

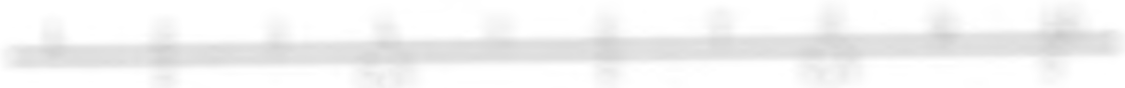
Remark: On 1 "prime"? The fact that mathematicians choose some type of symbols (in particular, of [illegible]) [illegible] approach. We consider several choices and finally pick one that seems most useful. [illegible] with [illegible], which is [illegible] [illegible] [illegible].

Any number about composite versus prime numbers? Several answers may be suggested. A composite number may be "simplified": 12 = 3 · 4, while 7 must stay as it. We can simplify 12 further: 12 = 3 · 2 · 2. We are heading into the fundamental theorem of arithmetic.

Multiplication is easy, writing numbers in factored form. But [illegible] that addition is difficult. How [illegible] [illegible]? [illegible] numbers are [illegible]?

Remark: Again, in constructing $\sqrt{2}$, what sort of properties are [illegible] used: divisibility? But we consider the relation to the fundamental theorem. What can [illegible]? [illegible] [illegible] [illegible]? Students may find general theorems.

Some numbers are divisible by 2, by 4, but we look at the pattern:



We see a fact that may be expressed in several ways:

a. All numbers divisible by 4 are divisible by 2.

b. Those numbers divisible by 4 form a subset of those divisible by 2.

[illegible]

[illegible]

[illegible]

[illegible]

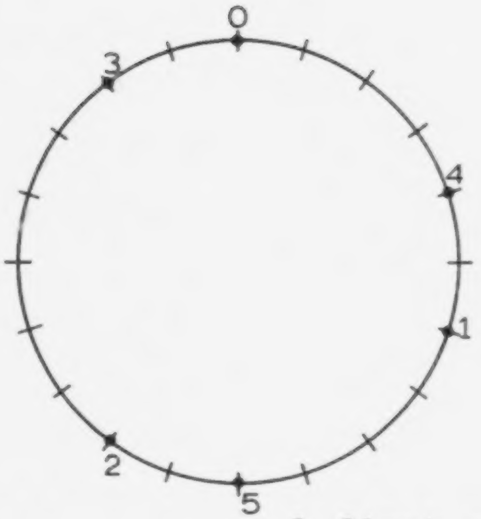
[illegible]

[illegible]

[illegible]

[illegible]

[illegible]



1=first step
2=second step

ask different class members to experiment with different n and a , and try for conclusions.

Presenting the conclusions so far, the gcd of n and a appears. Apart from this, various ways of proving facts occuring here may be suggested. For instance, a more or less geometric proof that the final spacing is even may be worked out. This is certainly a good topic for discussion in class. Later a more algebraic proof may be given. Now the two proofs may be compared. The fact that if there is a short space between two dots, all other spaces will be equally short has its analogue in equations. The two proofs may be written side by side, showing their final identity.

Having marked the "4" dot, how may we show that we have already gone once around the circle? Different ways may be suggested. In particular, the pattern may be shown along a line with multiples of 20 marked. Here residues mod 20 appear. In the first segment of length 20, the least residues appear. Equivalence (mod 20) and partitions come in naturally (if enough time is spent on the topic). The complete pattern shows the ideal generated by n and a ; the expression of (n, a) in terms of n and a is found. The usefulness of negative numbers is also apparent. Finally, if n is prime, we obtain divisibility properties concerning primes. From here the proof of the fundamental theorem is rapid. Comparison with a former proof is in order.

II. Multiplication of Negative Numbers

Having experimented with addition and subtraction of positive and negative integers, let us try multiplication. Here is one type of experiment that should be carried out: First, mark values of $x \cdot y$ in a plane with the usual coordinates. Start with non-negative x and y ; use just integers. Now look for

-12	-9	-6	-3	0	3	6	9	12	15
-8	-6	-4	-2	0	2	4	6	8	10
				0	1	2	3	4	5
				0	0	0	0	0	0
							-3		
							-6		

some other values. The pattern certainly suggests negative results. Why should we choose, for instance, the top row shown? From 9, subtracting 3 gives 6; subtracting 3 now gives 3; next, 0; next, -3; etc. This makes a simple pattern. It now has the property: From -6, adding 3 gives -3; adding 3 gives 0; adding 3 gives 3; etc. Let us state this in formulas: Adding 3 to 3k gives 3(k + 1), or

$$3(k + 1) = 3k + 3 .$$

How about bigger steps to the right? Soon the distributive law appears. Thus desiring the simple pattern is equivalent to desiring the distributive law. Note that a simple pattern appears first, use of a law later.

Now how about filling in the lower left-hand part? Clearly we desire positive numbers to keep the symmetry. We have found the product of negatives.

Since many feel that negative times negative should be negative, let us try this pattern also. Keep both patterns handy, and compare the two while working in various ways with integers (or real numbers). Moreover, a model with product equal height with both definitions (two models) is useful. One has sharp edges in the second model and has lost nice symmetry.

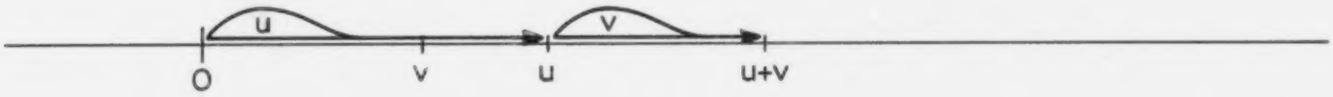
III. Directed Lengths

Suppose we have a line with a starting point and a direction along it. We have also a kit giving or manufacturing arrows or "vectors" which may be laid on the line, each in its given direction, from the starting point 0 or from certain other points. We shall study the resulting system. The application to measurement along a line (without a starting point) will be clear enough; we do not consider this here.

With any vector in our kit, named u for instance, laid with its "start" at 0, mark its "end" with the symbol u. This gives a picture of a vector from the kit.

Let us assume we can add:

a. Given u and v from the kit, we can manufacture $u + v$; v may be laid after u , giving the same mark as $u + v$ laid from the start.



The construction makes clear that we have (or require):

b. Given u, v, w from the kit, we have $(u + v) + w = u + (v + w)$.

Assume the kit contains a knife point with handle:

c. There is a vector 0 such that $u + 0 = 0 + u$ for all u.

Assume a vector can be turned around; rather, it can be copied backwards:

d. Given u, there is a vector $-u$ such that $u + -u = -u + u = 0$.

Finally, assume commutativity:

e. For all u, v we have $u + v = v + u$.

We can of course mention "group," etc.

Let us now assume order on the line or between the points marked on the line. (With the assumption about the points marked we need not worry about the order being Archimedian.)

f. The marks are simply ordered: if $u > 0$, then $u + v > v$ for all v .

Now we ask, "What is the system like? What is the set of all possible marks on the line like?" One soon sees that there are two cases?

Case 1. There is a smallest positive vector. In this case it is easy to see (class experimentation) that if we call this vector a , for instance, that

$$a,\ a + a,\ a + a + a,\ \ldots,\quad -a,\ -a + a,\ \ldots$$

is the set of all vectors in our kit and gives all the marks on the line.

Let us devise a special notation.

$$(\text{'})a = a, \quad (\text{''})a = a + a, \quad (\text{'''})a = a + a + a,$$

etc. Also some similar notation for negative vectors. We now have

$$(\text{'''})a + (\text{''})a = a + a + a + a + a = (\text{'''''})a\ ,$$

which suggests defining, among our new symbols,

$$(\text{'''}) + (\text{''}) = (\text{'''''}).$$

We then have the distributive law:

$$na + ma = (n + m)a, \quad \text{all}\ \ n, m,$$

in fact, as is easily seen by using u in place of a .

Note that we have *discovered* the group of integers and have found how this group operates on our kit of vectors.

Remark: We may prove the commutative law in this case.

Case 2. There is no smallest positive vector.

In this case it is interesting first to consider what we can do with just two vectors. We permit ourselves to construct others from these two. It may happen that we may form $ma + nb$, obtaining 0 (with none of a, b, m, n, zero). This brings us back to an earlier topic; also it may be used to introduce rational numbers as operators on our kit. If there is no such relation, the class may experiment with what may be done with the two vectors. We discover that we may manufacture arbitrarily small ones and are thus necessarily in Case 2.

Consider Case 2. We see that the marks on the line are dense. This suggests assuming completeness. For instance, assume mechanically that we may lay any number of vectors from 0 together and push a block down from the right till it can go no further and then mark the point reached.

Through our mechanical model, consider now our possible operations. Choose those vectors u which (with some fixed a) have the property $u + u + u < a$. Lay all these from 0, and push the block against them, giving v. It is quite easy to see that $v + v + v = a$. Thus we may "divide" a vector by ('''). Now all rational numbers operate on our kit. By completion, all real numbers operate. We have now constructed the real number system and its operation on our kit.

Summing up, the theory of directed lengths on a line is the theory of a simply ordered set allowing the real numbers as operators, or the theory of a one-dimensional vector space. We have accomplished several things: We understand better the operations of measurement in one dimension; we see how the real numbers come in of necessity to measurement.

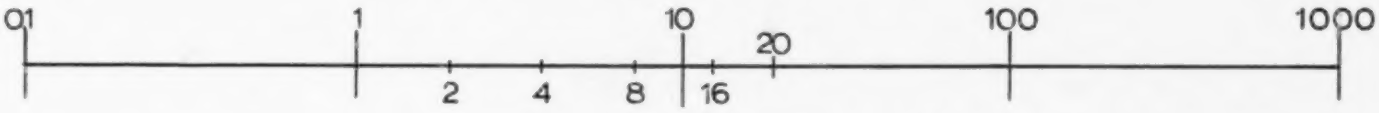
Note that if we choose a fixed vector $a \neq 0$ and express all vectors as multiples of a, we obtain an isomorphism of the real numbers with the system of vectors. In particular, any vector equals x inches and also equals y feet. A true equation is

$$1 \text{ ft.} = 12 \text{ in.}$$

For some peculiar reason, many present-day texts fear to use lengths as mathematical objects and are thus forced to say 1 foot "measures" the same as 12 inches. Worse, one is supposed to write equations in numbers only, making difficult the keeping track of actual vectors involved (especially when changing "units").

IV. Logarithms

Having worked with a "slide rule for addition," let us try a "slide rule for multiplication." First, choose it long enough to include 0.1, 1, 10, 100, 1000, say. Now try marking 2. From this we find 20. Also mark 4, 8, 16. We already have a test for our choice of 2: Does 16 seem to be at a nice point between 10 and 20? If not, move 2 a little. Now try some other numbers and resulting factors.



We noon note a nice property of the decimal system: When we mark 2, we may at once mark 0.2, 20, 200. Why do all these separately?

Recalling our study of points on a circle, let us construct a circular slide rule. Here 0.1, 1, 10, etc., will all be at the same point. With a carefully made cardboard rotating circle and a good manner of marking and erasing, a group of people can soon construct a rather accurate rule. Playing with these rules, properties of logarithms have an immediate clarity, and the shape of the logarithm function (transferred to a graph) has a strong reality.

ON THE ROLE OF LOGIC IN ELEMENTARY MATHEMATICS

Hassler Whitney

We shall consider here the question of what logic is needed in the study of elementary mathematics, say, through algebra and what and how such logic may best be taught. To get at the heart of the matter, suppose the following complaint is made during a class:

"I don't really understand what is going on."

Somewhere there has been a breakdown in communication so that the student cannot follow the reasoning process. The real issue is that of the reasoning process and, more generally, of the communication of the ideas. We shall use the term logic to cover this. For instance, in studying solutions of quadratic equations, the use of "variables," the question of whether a given "equation" is something you are supposed to solve or is an identity or defines a function, and what is being proved versus what is being discussed will be puzzling to the student. All such matters must be considered in studying the logical structure of an exposition.

Since carrying on mathematics consists essentially in carrying on precise reasoning, there is no question that logic must be considered in the teaching of mathematics. One extreme position is to hold that the student intuitively comes to understand what is going on and learns the underlying reasoning process through actual use. This is the classical position, which certainly has had a large degree of success. However, one must also realize that it can also fail to a large extent (see the quote above). At the other extreme, one can give a course in logic perhaps especially for mathematics students, say, in a formal axiomatic manner. The result is apt to be simply a new mathematics subject with little actual relation to other math courses. (One recent text omits the existential quantifier as being too complicated for a first course.) In between, a text may contain a chapter on logic; it most likely studies propositions through truth tables with unreal applications to real life and has some discussion of quantifiers. Uses of symbols (a start towards good communication) may appear. Again the chapter is usually largely forgotten in the rest of the book.

So far, no good solution is in sight. So to start, let us look at a few statements occuring in the grades and see what formal logical elements are involved.

a. You can subtract b from a, provided that $b \leq a$. If ..., then $\exists$, ...

b. If 4 will go into n, so will 2 $(\forall n)$. If P_n, then Q_n. (Actually, existential quantifiers are also involved if one used the definition of devisibility.)

c. If 2 is divisible by n and 3 is divisible by n, then 6 is divisible by n. $(\forall n)\ P_n \& Q_n \Rightarrow R_n$.

d. None of 2, 3, 4, 5, 6 go into 7. Hence 7 is prime. Not $\exists$ ..., hence ....

There is certainly no difficulty in the student's grasping the intuitive meaning, in fact the precise meaning, of the above statements. Thus the elements of logic in the restricted sense are well within the student's power. When the logical elements get more complex, it becomes worthwhile to analyze the logical structure. To this end the structure should have been examined to some extent earlier, and the general problems of communication should come commonly to the fore.

We suggest now a program for the "teaching" of logical concepts during the general mathematical studies. The division into grades is rather arbitrary, given for the sake of some kind of outline.

Early grades. The notions "true," "false," will arise; also "for all," "there exist." Certainly one will distinguish between all, some, none. Just how many is being more specific yet. Implication in its general sense appears; see a above, for instance.

Grade 7. Here is a good time to give some actual number theory; some examples of statements are given above. One can examine to some extent the logical notions involved without introducing logical notations other than for momentary shorthand notations. Students can give alternative formulations of statements. Those may be written out and compared. Thus one has a real start toward equivalence of logical expressions. This may be continued in grade nine.

Grade 10. So far we suppose that symbols have been used in the very concrete sense as names of numbers (or other things). For example, one may have done some elementary work in solving equations.

Suppose we have been given a number, which we may call x. We are told that $5x - 2 = 2x + 13$. Can we find this number? From the meaning of the equal sign, we know that we may add the same number, say, 2, to both sides. Thus $5x - 2x = 13 + 2$, and $3x = 15$. The only answer to this multiplication problem is 5. Thus x must be 5. (We may now check to see if 5 is really such a number.)

Note that there has been no use of "variables"; symbols are used in their normal manner. Through further such examples one may compare the meaning of equivalences and implications.

In the further study of solutions of equations, "solution sets" arise. If we write down a series of equations, as would be done in solving an equation as above, one may ask a separate question. Consider each equation separately without being told that x was some definite number. Then each equation is not a statement but a pattern with which we associate a solution set. What is the relation between the equations? The answer is obtained mathematically by almost the same procedure as in the example above, but the meaning is now logically much more involved. The difference between these meanings should be carefully considered.

Students should be given some help in reading and understanding the texts. In particular, they will begin to realize that there are both assertions, with strict mathematical meaning, and discussion, which may be more vague in character. For example, "the equation" may have several meanings, and one need not look for it precisely. Rather, they should pull out the mathematical content of what is said.

The meaning of a statement as presenting information arises. For example, we are told first that x is a number and that

$$x^2 + x - 6 = 0 . \tag{1}$$

What do we know about x ? From (1) we derive

$$x = 2 \quad \text{or} \quad x = -3 . \tag{2}$$

Now we learn that

(3) $$x > 0 .$$

It follows that $x = 2$. The information contained in (3) is larger than that in (2).

Note that after having learned (2), we may say

(4) $$\text{If } x > 0, \text{ then } x = 2.$$

This seems extraordinary. Frim the small amount of information $x > 0$, we obtain the precise information $x = 2$. This is so, since we are assuming all the time that (1) holds. Thus the meaning of a statement depends on what has been said before.

Consider now the statement

(5) $$\text{If } x \neq 0, \text{ then } \frac{x}{x} = 1.$$

This is obvious; if we know that x is not 0, then we may form $\frac{x}{x}$ and get the answer 1. The general interpretation of "if ..., then..." is needed; the definition from truth tables will not do.

It is apparent that one is getting deeper into logical concepts. However, at all times what is actually happening should be thoroughly understood. Discussions of the logic merely confirm the understanding.

Higher grades. It may be useful to work with mathematical statements expressed in logical form to some extent to see the picture clearly and to help understand the workings of logic. For instance, suppose we guess that every number divisible by both 6 and 21 is also divisible by 126. Later we think this is not so. How do we express this? There is a number divisible by 6 and 21 but not by 126. We find such a number, in fact, say, 42. Logically, we have seen the equivalence between the statements

$$\text{not } -(\forall n)(6 \text{ div } n \text{ and } 21 \text{ div } n \Rightarrow 126 \text{ div } n),$$
$$(\exists n)(6 \text{ div } n \text{ and } 21 \text{ div } n \text{ and not } 126 \text{ div } n).$$

In particular, we note the manner of negating the implication, provided that both parts of the implication are statements in advance (unlike in (5)).

As in all parts of mathematics, one introduces a notion best when it is needed. Implication, for instance, should not be introduced through a definition with examples like: If $1 = 2$, then $3 = 3$. There is no point in defining this. In "if $x > 5$, then $x^2 > 25$" there is some real point in a definition. (Moreover, the best definition is: This tells us nothing if x is not > 5; and this definition has general application, which the truth tables do not.) Is there ever any need to introduce truth tables? The following may give some reason for this.

Suppose we have learned that

$$\text{If } ab = 0 \text{ and } a \neq 0, \text{ then } b = 0 .$$

We would like to prove

$$\text{If } x^2 = 0, \text{ then } x = 0.$$

Of course the best proof is to note that if x is not zero, we know that x^2 is not 0. But nevertheless, let us use the former statement. Setting a and b equal to x (say x is a given number), we have

$$\text{If } x^2 = 0 \text{ and } x \neq 0, \text{ then } x = 0.$$

We seem to need an extra hypothesis, that x is not 0; but certainly we cannot believe this. We can reason: Suppose x were not 0, etc. But this is strange enough that it calls for a very clear discussion of truth values. The truth tables may help here.

We do not suggest going into a formal treatment at any point, except in a separate course for those interested. For instance, consider the following:

We see that if 6 div n, then 2 div n. Also, if 6 div n, then 3 div n. We may state this as follows: If 6 div n, then 2 div n and 3 div n. If there is any possible question, let us test this: Suppose 6 div n; we wish to show that 2 div n and 3 div n. We know that if 6 div k, then 2 div k for all k. Hence using this for our number n, 2 div n. Similarly, 3 div n.

Now let us see what a formal proof requires. The required formula is:

$$(\forall n)(6 \text{ div } n \Rightarrow 2 \text{ div } n) \;\&\; (\forall n)(6 \text{ div } n \Rightarrow 3 \text{ div } n)$$
$$\Rightarrow (\forall n)(6 \text{ div } n \Rightarrow 2 \text{ div } n \;\&\; 3 \text{ div } n) \,.$$

A formal proof in almost any system is sure to be long and complex, certainly hiding the inherent extreme simplicity of the result.

We make one final remark to illustrate the meaning of a statement. We wish to show that $\sqrt{2}$ is not rational. If it were, we could write $a^2 = 2b^2$, for some smallest pair of integers a, b. Since a^2 is even, we conclude that a is even: $a = 2c$. Now we have $b^2 = 2c^2$, in contradiction to the choice of a and b. Hence $\sqrt{2}$ is not rational. We note two things: In working with a and b, since we have actual integers before us, we know how to use them. In the end we found that in reality there were no such animals. How explain that we worked with them? All our statements had meaning and were true under the hypothesis that $\sqrt{2}$ was rational, not separately from this. Note also that in the proof our statements were both true and false, a contradiction we became aware of later.

THE USE AND IMPORTANCE OF DEFINITIONS IN MATHEMATICS

H. O. Pollak

One of the criticisms of curriculum reform in mathematics which is commonly heard is that the material has become too formal. It is claimed that the use of simple intuition has been replaced by reference to definitions and strict deductions from these definitions, and that this is not in the spirit of good mathematics pedagogy. I don't want to deal with the question of the validity of this criticism, but to make a comment on the mathematical point which it involves.

The fact is that definitions mean many different things to mathematicians at different times, and that definitions are used in very different ways. I can best illustrate my point by some examples.

A. Consider the problem of defining inverse trigonometric functions. A mathematically satisfying attack on defining arc sin X is to list all the properties that you would like this function to have. You find that it is impossible to have all these properties simultaneously; that is, they are internally inconsistent. You therefore have to give something up, and you finally choose the definition which appears to be most consistent with the calculus. Once you have made this definition, you rigorously stick to it and rarely go back to the intuition on which the definition is based. The reason for this is that the intuition is unsafe. The only way you will compute correctly with inverse trigonometric functions is to work very carefully with the branch that is consistent with the calculus.

B. An example at the opposite end of the scale is the definition of an ordered pair from the notion of an unordered pair. This is an exercise of mathematical showmanship to prove that you are clever enough to use the set notion in order to define an ordered pair. You will certainly never make use of this definition in working with ordered pairs.

C. Most examples, I am sure, fall somewhere between these two ends of the spectrum; that is, we will sometimes use the formal definition and sometimes the intuitive background to the formal definition. The formal definition of a function falls into this category. We first build up an intuition for the notion of a function from considering different ways to define it. Thus, we may use a picture, or a graph, or a verbal description, or a table or a formula, or a machine (where you imagine putting a number in at one end and another comes out at the other). From such examples and many more, you build up an intuition for the notion of a function and finally make a formal definition as a collection of ordered pairs. Later on, when you work with functions, you sometimes use the definition and sometimes return to one or more of the intuitive pictures which are behind the definition. Thus, if you want to be careful in distinguishing between a relation and a function, you may very well find considerable pedagogical value in the formal definition. On the other hand, if you want to discuss the notion of a function, you will probably find the intuitive machine picture easier than any other way. You just think of the output of one machine as the input of the next.

We must not make the mistake in our curriculum materials of assuming that definitions will be used in only one way. If a definition always supplants the intuition which leads to it, this may result in excessively dry materials. If a definition never supplants the intuition which leads to it, it is pretty useless. We must be honest with the students and let the mathematical abstractions take over in any of the variety of ways which might be most natural to the particular problem.

UNINVITED COMMENTS ON THE DEFINITION OF FUNCTION

G.S. Young

This is a footnote to Henry Pollak's wise remarks about definition, but addressed to the definition of function as a collection of sensed pairs.

In our book, Hocking and I give the definition the full treatment: A function $f : A \to B$ is a triple (A,B,G), where A and B are sets and G is a collection of ordered pairs (a,b) such that the first element of each pair is in A, the second element of each pair is in B, and each element of A is the first element of one and only one pair of G.

One could comment, and I would not argue on logical grounds, that A is not necessary; it could be defined by taking the union of the first elements of pairs of G. But this is unimportant. In topology, (and in some other places) there is real reason for signalling out the set B. With A fixed, and with the same collection of sensed pairs, changing B may really change properties of a function. Take for example $A = \{(x,y) | x^2 + y^2 = 1\}$, $B = B^2$, $B' = B^2 - (0,0)$, and let G be the collection of pairs $((x,y),(x,y))$. That is, each point of A is paired with itself. Let $f : A \to B$, $g : A \to B'$ be defined by this collection. Then, f is deformable to a point (in B), and g is not deformable to a point (in B'). We say f and g are different functions because of the very practical reason that they have different properties. Here the full notation and definition $f : A \to B$ is really useful. You need the fine distinction.

In mathematics through calculus, the set A, the domain, needs to be carefully specified (even though it can be reconstructed from G), but B is not terribly important, so long as it is big enough to contain all of the second elements of pairs in G. One can make up properties of elementary functions that change when one changes B, but they are not ones that cause confusion. To take one example, let f be the function that assigns to each real number x the number x^2. One set that I could use for B is the set of real numbers. With this B, it is true that B contains an open set that is entirely contained in the image of A; for example, the non-negative reals. Another set I could use for B is the set of all complex numbers. Then, with this B it is not true that B contains an open set entirely contained in the image of A. If you feel you want to emphasize this sort of point, then you can make a fuss about $f : A \to B$, in an early course. Otherwise, I now prefer to say, "A function f a set A into a set B is a collection of ordered pairs, etc.", playing down A and B, and always calling it f above.

I do want to keep the ordered pairs, not for logical reasons, but for pedagogical reasons. (1) I believe the definition emphasizes the single-valuedness better than any other. (2) It emphasizes the fact that you really do not want a formula. You can explain how much freedom you have in defining a function easiest by considering an element a and pointing out that it is just for that one a that you have to decide on the second element of the pair (a,__). (3) It seems the easiest way to get across that A and B need not be sets of numbers, that in particular, in real life A need not be a set of numbers. (4) There are certain points of precision that come across best in this framework. Consider the following: Given the relation -- if you want the term -- $x^2 + y^2 = 1$. How many functions on $[-1,1]$ are defined by it? It is a very bright freshman who ever says anything other than two for the answer. The point is, of course, that for each x you have the choice of pairing with it either $+\sqrt{1 - x^2}$ or $-\sqrt{1 - x^2}$, and clearly can get uncountably many different functions. Continuous functions? That's a different functions. Continuous functions? That's a different thing. One has the same thing on the inverse of $y = x^2$.

It is for reasons like these that I like ordered pairs. It is only when I want to emphasize such points that I stick with the definition of ordered pairs. Otherwise, I see no harm in saying, "Let f be defined by $f(x) = x^2$ for all real numbers x", or even, "Let $f(x) = x^2$".

I agree thoroughly with Pollak's desire to use all sorts of ideas for functions. The Begle meatgrinders (in his Calculus) are wonderful for composition of functions, for example.

There are all sorts of mistakes that sensed pairs do not help you avoid. The minimum of $y = x^{2/3}$ is at $x = 0$. No amount of sensed pairs will keep some freshmen from saying $y' = 2/3x^{-1/3}$, and setting $y' = 0$ and concluding there is no minimum. Here the function as a graph is the best approach. Incidentally, was anyone bothered by "$y = x^{2/3}$"?

ON THE SETTING AND FUNCTION OF SETS AND FUNCTIONS

Leonard Gillman

Mathematicians tend not to care what an object is but only what its properties are.

(1) A cardinal number is an object (thing, entity, set, element, gizmo) associated with a set in such a way that two sets have the same cardinal number if and only if they are equipotent.

One can now construct the theory of cardinals.

(2) An ordered pair (a,b) is an object (thing, entity, set, element, gizmo) characterized by the following definition of equality: $(a,b) = (a',b')$ if and only if $a = a'$ and $b = b'$ (the meanings of the last two equalities being already known).

One can now proceed to work with ordered pairs.

(3) We did not say what a cardinal number is nor what an ordered pair is but only what their characteristic properties are. Those who work in logic and foundations may supply definitions in terms of prior notions. For example, in terms of sets, we have:

Definition. $(a,b) = \{\{a\}, \{a,b\}\}$.

Theorem. $(a,b) = (a',b')$ if and only if $a = a'$ and $b = b'$.

The first thing we do is prove the theorem and then we never again have to refer to the definition.

(4) The proof of this theorem is a good example of some set theory just beyond the level of triviality of the popular school-math set theory. For this reason I urge anyone who may not happen to be familiar with it to sit down right now and work it out. The prior information needed is the definition of equality for sets ("axiom of extensionality"): two sets are equal if and only if they have the same elements. In the proof, one can argue by counting elements, but I consider that inelegant and urge its avoidance.

I conjecture that a lot of new-math authors left out the definition of ordered pair simply because they (fortunately) did not know of it.

(5) Before deciding on a definition of function, first consider seriously whether it ought to be defined at all. If so, consider carefully the uses to which it will be put and then choose the definition most naturally suited to them rather than whatever yields the quickest derivations of formal properties. I have little doubt that the definition as a set of ordered pairs is the worst one.

(6) First, another example to suggest that a function from A to B means more than just a subset of $A \times B$ (of a particular kind). Recall that X^Y stands for the set of all mappings (functions) from Y into X. The following paragraph appears on page 140 of my book with Jerison:

> Let φ be a given mapping from a set A into a set B. For each mapping g from B into a set E, the composition $g \cdot \varphi$ carries A into E. Thus, φ induces a mapping $\varphi' : E^B \to E^A$; explicitly,
>
> $$\varphi' g = g \cdot \varphi .$$
>
> There is a duality between the properties one-one and onto (provided that E has more than one element): φ' is one-one if and only if φ is onto, and φ' is onto if and only if φ is one-one. The verification of these facts is left to the reader.

Evidently, if we augment the range of φ, we alter the domain of φ'.

The second half of the quotation is included as another candidate for the borderline of sophistication. Once more, everyone is invited to assess the level. The parenthetical proviso is needed in just one of the four parts.

(7) In seeking a definition of function, I do not worry about how many functions are defined by $x^2 + y^2 = 1$. I consider the problem unimportant. It is not used, needed, or referred to in any high school or college course -- mathematics or other. I'm for the chap who says the answer is two.

(8) I suspect that the best way to think of a function is as an association, i.e., as the process of associating, i.e., as the passage from a given element to its associated element. The emphasis is on the act of associating rather than on the totality of pairs of associates. Note the suggestiveness of the notation: $a \rightarrow b$.

(9) There are many important situations where this view is natural. (Maybe it always is.) For example, when Peano tells me that every natural number has a successor, I do not picture a great big set,

$$\{(1,2), (2,3), \ldots\},$$

but rather the passage

$$n \rightarrow n^+.$$

It is possible that I think in terms of (shudder!) variables.

(10) I wonder whether anyone really does think of sets of ordered pairs other than when picturing graphs. When you think of sin analytically, do you say things to yourself like

$$(\pi, 0) \in \sin?$$

I talk about solving the equation

$$\sin x = \cos x.$$

Is there anyone who thinks instead of specifying the set

$$pr_1, (\sin \cap \cos)?$$

(11) The "set theory" discussed in school mathematics is bringing increasing discredit to mathematics and mathematicians and should be discarded. Right now is a good time.

Here is another test problem: prove that the real line is the union of the intervals $[-1,1], [-2,2], \cdots, [-n,n], \cdots$.

Sets should be postponed until they can be introduced with significant content and applications -- for example, somewhere near the level of the theorems mentioned above.

(12) Back to functions. Let R be a set -- e.g., the set of real numbers. Consider the following renditions.

> [A] There exists a function $f : R \times R \to R$ (thus, f is a subset of $(R \times R) \times R$) such that, for all $a, b, c, u, v, x, y \quad R$,
>
> $$\begin{aligned} \text{if } & ((a,b), u) \in f, \\ & ((b,c), v) \in f, \\ & ((u,c), x) \in f, \\ \text{and } & ((a,v), y) \in f, \\ \text{then } & x = y. \end{aligned}$$
>
> (Before continuing, try to figure out what the hell that means.)
>
> [B] There exists a function $f : R \times R \to R$ such that, for all $a, b, c \quad R$,
>
> $$f(f(a,b), c) = f(a,f(b,c)).$$
>
> (Stop here too to dwell. Better than [A], eh?)
>
> [C] There exists a binary operation $\cdot$ defined on R such that, for all $a, b, c \in R$,
>
> $$(a \cdot b) \cdot c = a \cdot (b \cdot c).$$

References. SMSG working papers:

H. O. Pollak, The Use and Importance of Definitions in Mathematics, 29 June 1966.

G. S. Young, Uninvited Comments on the Definition of Function, 30 June 1966.

ON "ON THE SETTING AND FUNCTION OF SETS AND FUNCTIONS"

Gail Young

In a well-known series of school texts occurs a chapter on rational numbers, one page of which starts off with a section on multiplication. On the page is an example: $(3,1) \cdot (5,2) = (17,11)$. I have derived a great deal of innocent pleasure showing this to professional mathematicians and asking them what is going on, and not one has ever realized that this is $3 \cdot 2 = 6$, or, perhaps more fairly, $(+3) \cdot (+2) = +6$.

It turns out that in $(3,1)$, the "3" and the "1" are really equivalence classes of ordered pairs of whole numbers, and $(3,1)$ really is an equivalence loss of ordered pairs of these ordered pairs.

To me, there is exactly one reason for ever going through anything like this. Suppose that I have written down a set of axioms for some mathematical system. I want to know that the axioms are consistent, that I will never end up proving two contradictory theorems. I know from Godel's work that I can never hope to prove from the axioms themselves that I will have no contradiction. Also I cannot hope to prove all possible theorems and show no two are contradictory. What can I do? I can find some things whose existence I believe in already, and show that with proper definitions these satisfy the axioms. If the axioms were contradictory, the same contradiction would exist in the things I believe exist, and surely I can't believe in contradictory things, can I? (All this is discussed in the first chapter of R. L. Wilder's Foundations.)

If I believe in the whole numbers, I can construct a model of the integers (in some such way as this 8th-grade text, which really does it rather well), then of the rationals, then of the reals. Given the reals, I can construct the set of ordered pairs of reals, define "line" in terms of these, etc., and set a model of Euclidean geometry. From that model I can get a model of non-Euclidean geometry, etc., etc.

Every professional mathematician should see an example of such a treatment once in his training. Conceivably, the 8th grade may be the place? I don't believe it is, myself.

By this time the reader and I are both wondering what my point was in saying all this. The point, I think, is this. A teacher should be just as rigorous, as full of sensed pairs, sets, etc., as the situation needs. The discussion of the rationals, short of this one question of consistency, does not need all this apparatus. In fact, it hides the real situation, which to my mind is that the rationals form part of applied mathematics. That is, they are the first complicated structure set up to handle physical problems - subdividing pies, piles of wheat, etc. To get that across, and to get a clear understanding of why they behave the way they do is, to my mind, far better than any amount of equivalence classes or sensed pairs of equivalence classes.

One incidental remark. If anyone can show that kids taught rationals that way do better in mathematical subjects than kids who have spent the same time doing something else, I will withdraw all my objections. The test of curriculum development is irrational: what happened in the classroom, and in the next course.

But I was making a further point. What I think Pollak, Gillman, and I are all concerned with is the sort of question raised by the example. Here is some perfectly good, valuable mathematics. When does one teach it in the full form?

To go back to the rational numbers for a moment, that rational numbers can be described by ordered pairs of integers, and that the operations can be defined that way is an important concept to get some grasp of. It is what lets one realize that the rationals can also be looked at as part of pure mathematics. I would not regard it as at all a bad thing to put in a couple of days in the 11th or 12th grade explaining this.

Set theory. I wish Gillman had said a little more about set theory in school mathematics. I suspect we are in agreement. To my mind, the only place in school mathematics where set theory should occur in any formal sense -- that is, with operations of $\cup$, $\cap$, $\in$, $\subset$ fully used -- is in probability. And here I am actually only saying that this is the way I first understood what was going on and how I finally got to where I could work complicated problems involving combinations and permutations. I could go back to an underlying set-theoretic situation and think things out.

There are a number of places, however, where the terms of set theory, to my mind, provide a natural language for mathematical discussion. I don't know what Gillman wants to do about cardinal arithmetic, but to my mind, $2 + 4 = 6$ because if you take a set of 2 things and another set of 4 things and combine them you get a set of 6 things. Kids should understand this. (Incidentally, this is, of course, another example of applied mathematics.) The moment you attempt the least formalization, you run into the question of whether the sets are disjoint. You can put in the effort to discuss $A \cap B$, $A \cup B$, $\emptyset$, etc. Should you? Again, the answer is what happens in the classroom. Kids should understand that if you have 4 women and 5 Indians in a room you may have anywhere between 5 and 9 people. I don't know when I learned this, but certainly not formally in school. Would my mathematical education have progressed better if this had been made fully conscious at any early stage?

In geometry, a line is a set of points. Why say "A line is made up of points" or any other 19th century language when you can say it in a clear standard terminology? An angle is (1) "the figure determined by two straight-line rays with a common end point"; (2) "the union of two straight-line rays with a common end point". Either definition is fine, for telling you what an angle is. Each is simple. The first one (or modifications) seems vague to me in some ways.

For example, do you mean the area between the rays? It is not as precise as the second. The second teaches something about a mathematician's use of language that the first does not. But "Let A and B be two straight-line rays and let $A \cap B = \{x\}$, x a common end point of A and B. The angle AB is $A \cup B$". That's just bad writing. Nothing whatever was gained by the letters and symbols.

Back to functions. In (5), Lennie asks whether a function ought to be defined at all. I think this is an important thing to decide, and something that has not received enough discussion in school mathematics. One certainly wants a dictionary type of definition, the sort of thing that gives you an idea of what the concept is, but whether you want a real mathematical definition, I am by no means sure. The trouble with mathematical definitions is that you are apt to believe that the definition tells you what the thing really is. A rational number darned well is not an equivalence class of ordered pairs of integers -- though that is one aspect of rational numbers. If I want to prove with complete rigor everything about rational numbers, I might start with this definition because I can use it in my proofs. Do I want to prove the sort of things about functions that need a rigorous definition? In school mathematics? Certainly in much of advanced mathematics, one needs the full definition of my last paper, or something equivalent. If Lennie proposed a definition of function different from mine, but that would take care of his example in (6) as well as my definition, I am sure each of us could work with either one, and the advantages of one would likely be a matter of personal preference.

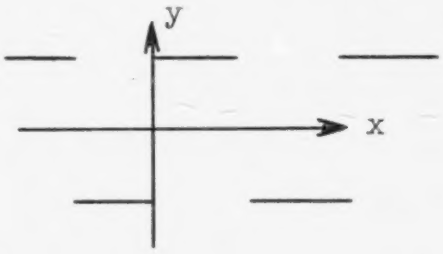
Perhaps one should not define "function", but give enough aspects of function so that every reasonable definition of function will occur implicitly. My own pedagogical manner would lead me to give the pair definition -- after motivation -- and doing all the others in terms of it. I have great difficulty in being deliberately vague in the classroom. If there is a clear way to say something precisely that the audience can understand at some level, I usually have to say it precisely.

For functions, I buy completely Lennie's (8), (9), (10) with the comment that I never think analytically of sin, but emotionally, and that what I say to myself is things like "Ooh, boy, look at that!" I'd never think of $(\pi, 0)$ as sin.

That leaves (7).

Statement 1. Consider the expression $x^2 = y^2$. How many functions does that determine? Even if we restrict ourselves to continuous functions the answer is 4, $y = x$, $y = -x$, $y = |x|$, $y = -|x|$. Only two are differentiable. The chap who says the answer is 2 in my problem will say the answer is 2 in this case. But the answer to his real, physical problem may be $y = |x|$; consider reflections, for example.

Consider $y^2 = 1$. That determines two continuous functions. An electrical engineer may, however, be interested in a square-wave function, like this. These things can really happen, in real problems. I think it important that the student's intuition be freed, sometime, for such things.



Statement 2. What I really want is for the student to be slapped down by a counter-example every time he leaves out a vital word in a hypothesis. I would like him to be scared silly every time he sees the word "function" with nothing in front of it like "continuous", "differentiable", "analytic". I rather like my example as a start on this traumatization. But I don't really care how many functions $x^2 + y^2 = 1$ determines. I'll bet I could cook up a "practical" problem where the answer is $y = \sqrt{1 - x^2}$, $x > 0$; $y = -\sqrt{1 - x^2}$, $x \leq 0$, though.

Statement 3. One thing that R. L. Moore makes conscious in his students is the tremendous importance of negative information in mathematics. "You need compactness in the hypothesis, because here's a counter-example". As a group, Moore students all spend much more time giving such examples than most people. I can't give a definition in class without giving quickly some examples of just what sort of pathology has sneaked in with the definition.

No one understands commutativity until they see examples of non-commutative structures, one thing that makes me wonder about the value of all the name dropping in K-6.

To summarize. If you are going to give a rigorous definition of function, really give it and really use it. What definition should be decided only after seeing what the results of trying to write up each approach look like. If not, the fact that functions can be defined by sensed pairs should be brought out and used.

Shakespeare's summary: "Function is smothered in surmise." Macbeth, Act 1, Sc. 2. [verbal communication from Warren Stenberg.]